LARGE CARDINALS AS PROPERTIES OF SHEAVES

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Abstract. We characterize several large cardinal notions as properties of sheaves.

1. Sheaves

Let X be a topological space, we denote by $\mathscr{O}(X)$ the set of all open subsets of X. $\mathscr{O}(X)$ can be seen as a category where the objects are the open sets and the morphisms are the inclusions of open sets.

We recall that a *sheaf on* X (a *Sets*-valued sheaf on X) is a contravariant functor F from $\mathscr{O}(X)$ to *Sets* -the category of sets- that assigns to each open set U of X a set $F(U)$, and to each pair of open sets $V \subseteq U$ of X a morphism $F_{V,U} : F(U) \to F(V)$ in the category *Sets* (called the restriction morphism) such that:

(Gluing axiom) if $\{U_i\}_{i\in I}$ is an open covering of an open set U, and if for each $i \in I$ we pick $s_i \in U_i$ such that for any $i, j \in I$ we have $F_{U_i \cap U_j, U_i}(s_i) = F_{U_i \cap U_j, U_j}(s_j)$, then there exists a unique $s \in F(U)$ such that $F_{U_i,U}(s) = s_i$ for each $i \in I$.

When there is no ambiguity, we write $s|U$ for the restriction $F_{U,V}(s)$. For every open U, the elements of $F(U)$ are called *sections*; a section over X is called a *global section*. Sections $\{s_i\}_{i\in I}$ satisfying the condition of the Gluing axiom are called *compatible*, the unique section s whose existence is guaranteed by that axiom is called the *gluing* of the sections $\{s_i\}_{i\in I}$

Trough the notion of Grothendieck topology one can generalise this to any cathegory $\mathscr{C}$. The usual notion of covering is replaced by the more general notion of *sieve*.

Definition 1.1. *A* sieve S *over an object* c *of* $\mathscr{C}$ *is a family of morphisms of codomain* c *such that for all* f, g *in* S*, the composite* $f \circ g$ *is in* S.

Given a sieve S over c and a morphism $f : d \to c$, the *pullback of* S *along* f, denoted f^*S, is the sieve on d given by left composition with f, i.e.

$$f^*S = \{g : e \to d;\ fg \in S\}$$

Definition 1.2. *A* Grothendieck topology J *on* $\mathscr{C}$ *is a collection of distinguished sieves* $J(c)$, *for every object* c *of* $\mathscr{C}$, *called* covering sieves *of* c, *such that*

1. *(Identity) the maximal sieve* $M_c := \{f; cod(f) = c\}$ *is in* $J(c)$;
2. *(Base change) for every sieve* S *in* $J(c)$ *and every morphism* $f : d \to c$, *the pullback* $f^*(S)$ *is in* $J(d)$;

(3) *(Local Character) for every sieve* $S \in J(c)$ *and every sieve* T *over* c, *if we have* $f^*T \in J(\mathrm{dom}(f))$ *for all* $f \in S$, *then* $T \in J(c)$.

A *site* on $\mathscr{C}$ is a pair $(\mathscr{C}, J)$ where $\mathscr{C}$ is a category and J is a Grothendieck topology on $\mathscr{C}$.

Definition 1.3. *GIven a site* $(\mathscr{C}, J)$, *a* J-sheaf on $\mathscr{C}$ with values on *Sets is a contravariant functor* F *from* $\mathscr{C}$ *to* Sets *that satisfies the gluing property, namely for every sieve* $S \in J(c)$ *and for every family of sets* $\{x_f\}_{f \in S}$ *such that* $x_f \in F(dom(f))$ *and* $x_{f \circ g} = F(g)(x_f)$, *there exists a unique object* $x \in F(c)$ *such that* $F(f)(x) = x_f$ *for every* $f \in S$.

2. Large Cardinals

We will consider two large cardinal notions, namely weakly compact and strongly compact cardinals.

Definition 2.1. *For a cardinal* κ,

(1) κ *is* weakly compact *if and only if, for every collection* T *of sentences of the language* $\mathscr{L}_{\kappa,\kappa}$ *with at most* κ *non-logical symbols, if* T *is* κ*-satisfiable, the* T *is satisfiable;*

(2) κ *is* strongly compact *if and only if, for every collection* T *of sentences of* $\mathscr{L}_{\kappa,\kappa}$, *if* T *is* κ*-satisfiable, then* T *is satisfiable.*

These notions admit a combinatorial characterisation in terms of properties of trees or similar objects. We recall the definition of the tree property.

Definition 2.2. *Given a regular cardinal* κ,

(1) *a* κ-tree *is a tree of height* κ *all of whose levels have size less than* κ;

(2) *we say that* κ *has the* tree property *when every* κ*-tree has a cofinal branch.*

Theorem 2.3. *(Erdös Tarski 1961) A regular cardinal* κ *is weakly compact if and only if,* κ *is inaccessible and every* κ*-tree has a cofinal branch.*

The strong tree property concerns special objects that generalise the notion of κ-tree, for a regular cardinal κ.

Definition 2.4. *Given a regular cardinal* $\kappa \geq \omega_2$ *and an ordinal* $\theta \geq \kappa$, *a* (κ, θ)-tree *is a set* F *satisfying the following properties:*

(1) *for every* $f \in F$, $f : X \to 2$, *for some* $X \in [\theta]^{<\kappa}$

(2) *for all* $f \in F$, *if* $X \subseteq \mathrm{dom}(f)$, *then* $f \restriction X \in F$;

(3) *the set* $\mathrm{Lev}_X(F) := \{f \in F;\ \mathrm{dom}(f) = X\}$ *is non empty, for all* $X \in [\theta]^{<\kappa}$;

(4) $|\mathrm{Lev}_X(F)| < \kappa$, *for all* $X \in [\theta]^{<\kappa}$.

As usual, when there is no ambiguity, we will simply write Lev_X instead of $\mathrm{Lev}_X(F)$. In a (κ, θ)-tree, levels are not indexed by ordinals, but by *sets of ordinals.* So the predecessors of a node in a (κ, θ)-tree are not (necessarily) well ordered and a (κ, θ)-tree is not a tree.

Definition 2.5. *Given a regular cardinal $\kappa \geq \omega_2$, an ordinal $\theta \geq \kappa$ and a (κ, θ)-tree F, a* cofinal branch *for F is a function $b : \theta \to 2$ such that $b \restriction X \in \mathrm{Lev}_X(F)$, for all $X \in [\theta]^{<\kappa}$.*

Definition 2.6. *Given a regular cardinal $\kappa \geq \omega_2$ and an ordinal $\theta \geq \kappa$,*

(1) (κ, θ)-TP *holds if every (κ, θ)-tree F has a cofinal branch;*
(2) *we say that κ satisfies the* strong tree property *if (θ', μ)-TP holds, for all $\theta' \geq \kappa$;*

For a more extensive presentation of this property, the reader can consult Weiss Phd thesis [**?**].

Theorem 2.7. *(Jech 1973, Di Prisco, Zwuicker 1980) A regular cardinal κ is strongly compact if and only if κ is inaccessible and the strong tree property holds at κ.*

3. Weak compactness and Sheaves

If κ is an ordinal, then we can see κ as a topological space whose open sets are the ordinals $\alpha \leq \kappa$. Then a sheaf on κ is a contravariant functor from κ to *Sets* with the localoty and gluing property.

First of all, note that for an ordinal κ, the category $(\kappa)^{op}$ corresponds to the category whose objects are the ordinals below $\kappa + 1$ (these are the open subsets of κ) and the morphisms are the inclusion maps.

Definition 3.1. *Given a regular cardinal κ, we say that a sheaf F on a topological space X is* κ-thin *when*

(1) *$F(U)$ is non empty, for every proper open set U;*
(2) *for each proper open set U, there are less than κ many extendible sections in $F(U)$.*

Thus, in the case of a sheaf F on κ, we have that F is κ-thin when $F(\alpha)$ is non empty for every $\alpha < \kappa$, and for every $\alpha < \kappa$, the set $\bigcup_{\beta > \alpha} F_{\alpha,\beta}[F(\beta)]$ has size less than κ.

Theorem 3.2. *Assume κ is a regular cardinal, the following are equivalent:*

(i) *κ has the tree property;*
(ii) *Every κ-thin sheaf F on κ has a global section*

Proof.

(i)$\to$ (ii) Let F be a κ-thin sheaf on κ. For every $\alpha < \kappa$, we let L_α denote the set of all extendible sections of $F(\alpha)$, equivalently $L_\alpha = \bigcup_{\beta > \alpha} F_{\alpha,\beta}[F(\beta)]$. We define a tree ordering $<_T$ on the set $T := \bigcup_{\alpha < \kappa} L_\alpha$ by letting

$$x <_T y \iff x \in L_\alpha,\ y \in L_\beta \text{ for } \alpha < \beta \text{ and } F_{\alpha,\beta}(y) = x.$$

It is easy to check that $(T, <_T)$ is a κ-tree. By assumption T has a cofinal branch b. Assume $b := \{x_\alpha\}_{\alpha < \kappa}$, then we have for every $\alpha < \beta < \kappa$, $F_{\alpha,\beta}(x_\beta) = x_\alpha$. By the gluing property, there exists a global section $x \in F(\kappa)$.

(ii)→ (i) Let $(T, <_T)$ be a κ-tree. We define for every $\alpha \leq \kappa$, $F(\alpha)$ as the set of all cofinal branches of $T \restriction \alpha$, and for $\alpha < \beta \leq \kappa$, we let

$$F_{\alpha,\beta}(b) := b \cap T \restriction \alpha.$$

We claim that F is a κ-thin sheaf. Given an ordinal $\alpha := \bigcup_{i \in I} \beta_i$ and compatible sections $\{b_i\}_{i \in I}$, there exists a unique gluing of the sections, namely $b := \bigcup_{i \in I} b_i$ which is a cofinal branch of $T \restriction \alpha$. For every $\alpha < \kappa$, we have

$$\{b \cap T \restriction \alpha;\ \exists \beta > \alpha\ (b \in F(\beta))\} = \{b \cap T \restriction \alpha;\ b \in F(\alpha+1)\} = F_{\alpha,\alpha+1}[F(\alpha+1)].$$

We observe that $F(\alpha+1)$ has the same size as $\mathrm{Lev}_\alpha(T)$, hence it has size less than κ, indeed, every branch in $F(\alpha+1)$ has a maximal element on $\mathrm{Lev}_\alpha(T)$. It follows that $|F_{\alpha,\alpha+1}[F(\alpha+1)]| < \kappa$. By the gluing property, there exists a global section $b \in F(\kappa)$. Then b is a cofinal branch for T.

□

Corollary 3.3. *Assume κ is an inaccessible cardinal, then κ is weakly compact if and only if, every κ-thin sheaf F on κ has a global section*

Proof. It follows from the characterisation of weakly compact cardinals in terms of the tree property. □

4. Strongly compact cardinals and sheaves

Let λ be any ordinal, we can see $\mathcal{P}(\lambda)$ as a category whose objects are the subsets of λ and the morphisms are the inclusion maps. Now assume κ is a regular cardinal and $\lambda \geq \kappa$. We define a Grothendieck topology J_κ on this cathegory as follows. For $X \subseteq \lambda$, we let $J_\kappa(X)$ be the set of all families $\{X_i\}_{i \in I}$ of subsets of X such that $\bigcup_{i \in I} X_i = X$ and for every $Y \subseteq X$ of size less than κ, there exists $i \in I$ such that $Y \subseteq X_i$.

Theorem 4.1. *Assume κ is an inaccessible cardinal, then κ is strongly compact if and only if, for every λ and every sheaf F on the site $(\mathcal{P}(\lambda), J_\kappa)$, if $F(X)$ is non-empty for every $X \in \mathcal{P}_\kappa(\lambda)$, then F has a global section.*

Proof. We use the characterisation of strongly compact cardinals in terms of the strong tree property.

($\Longrightarrow$) Assume κ is a strongly compact cardinal, hence κ is inaccessible and satisfies the strong tree property. Let F be a sheaf on $(\mathcal{P}(\lambda), J_\kappa)$ such that $F(X)$ is non empty for every X of size less than κ. Consider the union A of all $F(X)$ for $X \in \mathcal{P}_\kappa(\lambda)$ and let θ be the size of this set. We fix an enumeration $\{s_i;\ i < \theta\}$ of A and we let T be the set of all $< \kappa$-sequences of pairwise compatible sections in A. We show that T is a (κ, θ)-tree. Every element of T can be seen as a function $s : I \to 2$ over a set of indexes I of size $< \kappa$ such that $s^{-1}\{1\}$ is a set of pariwise compatible sections. By the inaccessibility of κ, every level $Lev_I(T)$ has size $< \kappa$ (because it is a subset of ${}^I 2$ that has size $< \kappa$). As κ satisfies the strong tree property, we can find a cofinal branch B for T. By construction, B

is a sequence of pairwise compatible sections. By the cofinality of B, we have that for every $X \in \mathcal{P}_\kappa(\theta)$ there exists $Y \subseteq X$ such that $B \cap Y$ is non empty. It follows from the sheaf property that $F(\lambda)$ is non empty (λ is the union of all its subsets of size $< \kappa$).

($\Longleftarrow$) It is enough to show that κ has the strong tree property. Let T be a (θ, κ)-tree where $\theta \geq \kappa$. We define a sheaf F on the site $(\mathcal{P}(\theta), J_\kappa)$ by letting $F(X) = Lev_X(T)$ for every $X \in \mathcal{P}_\kappa(\theta)$. Let f be a global section, then f must be a cofinal branch for T.

$\square$

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