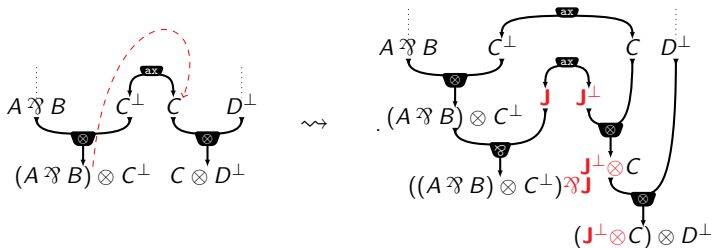


The Exponential Logic of Sequentialization



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TLLA 7th – July, 1st 2023

MLL proofs and nets

MLL $F = F \otimes F \mid F \wp F \mid A \mid A^\perp$

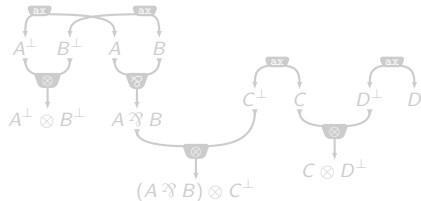
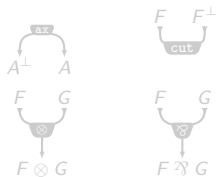
Proof trees

$$\frac{}{\vdash A, A^\perp} \text{Ax} \quad \frac{\vdash \Gamma, F \quad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta} \text{cut}$$

$$\frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \wp G} \wp \quad \frac{\vdash \Gamma, F \quad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G} \otimes$$

$$\frac{\frac{\frac{\frac{}{\vdash A^\perp, A}}{} \otimes \quad \frac{}{\vdash B^\perp, B}}{} \otimes}{\vdash A^\perp \otimes B^\perp, A, B} \wp}{\vdash A^\perp \otimes B^\perp, (A \wp B) \otimes C^\perp, C} \otimes \quad \frac{}{\vdash C^\perp, C} \otimes}{\vdash A^\perp \otimes B^\perp, (A \wp B) \otimes C^\perp, C \otimes D^\perp, D} \otimes$$

Proof structures



+ **Confluence**: if $R \Downarrow R'$ then R' is unique (contrary to proof trees)

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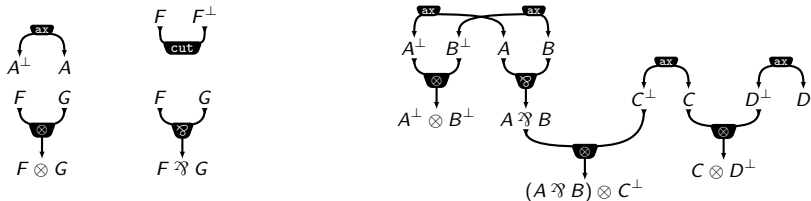
Proof trees

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$$\frac{\frac{\frac{\frac{}{\vdash A^\perp, A}}{\vdash A^\perp \otimes B^\perp, A, B} \otimes}{\vdash A^\perp \otimes B^\perp, A \wp B} \wp}{\vdash A^\perp \otimes B^\perp, (A \wp B) \otimes C^\perp, C} \otimes \quad \frac{}{\vdash C^\perp, C} \text{Ax}}{\vdash A^\perp \otimes B^\perp, (A \wp B) \otimes C^\perp, C \otimes D^\perp, D} \otimes$$

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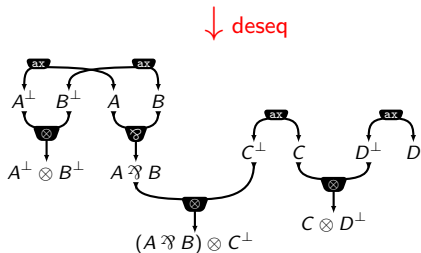
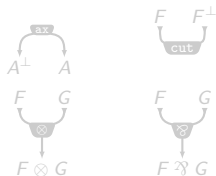
Proof trees

$$\frac{}{\vdash A, A^\perp} \text{Ax} \quad \frac{\vdash \Gamma, F \quad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta} \text{cut}$$

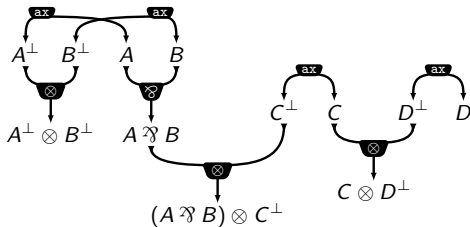
$$\frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \wp G} \wp \quad \frac{\vdash \Gamma, F \quad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G} \otimes$$

$$\frac{\frac{\frac{\frac{}{\vdash A^\perp, A}}{\vdash A^\perp \otimes B^\perp, A, B} \otimes}{\vdash A^\perp \otimes B^\perp, A \wp B} \wp}{\vdash A^\perp \otimes B^\perp, (A \wp B) \otimes C^\perp, C} \otimes \quad \frac{}{\vdash C^\perp, C} \text{Ax}}{\vdash A^\perp \otimes B^\perp, (A \wp B) \otimes C^\perp, C \otimes D^\perp, D} \otimes$$

Proof structures



– A proof structure is a **net** if it is the **desequentialisation** of some proof tree.

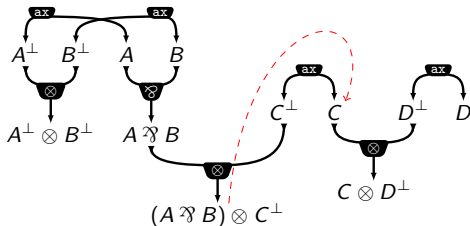
Retrieving some sequentiality: **Jumps**
 \swarrow seq \searrow

$$\frac{\frac{\frac{\overline{\vdash A^{\perp}, A} \quad \overline{\vdash B^{\perp}, B}}{\vdash A^{\perp} \otimes B^{\perp}, A, B} \otimes}{\vdash A^{\perp} \otimes B^{\perp}, A \wp B} \wp}{\vdash A^{\perp} \otimes B^{\perp}, (A \wp B) \otimes C^{\perp}, C} \otimes \quad \overline{\vdash C^{\perp}, C}}{\vdash A^{\perp} \otimes B^{\perp}, (A \wp B) \otimes C^{\perp}, C \otimes D^{\perp}, D} \otimes$$

$$\frac{\frac{\frac{\overline{\vdash A^{\perp}, A} \quad \overline{\vdash B^{\perp}, B}}{\vdash A^{\perp} \otimes B^{\perp}, A, B} \otimes}{\vdash A^{\perp} \otimes B^{\perp}, A \wp B} \wp \quad \frac{\overline{\vdash C^{\perp}, C} \quad \overline{\vdash D^{\perp}, D}}{\vdash C^{\perp}, C \otimes D^{\perp}, D} \otimes}{\vdash A^{\perp} \otimes B^{\perp}, (A \wp B) \otimes C^{\perp}, C \otimes D^{\perp}, D} \otimes$$

[1991] Quantifiers in linear logic II. Girard.

[2008] Proof nets sequentialisation in multiplicative linear logic. Di Giamberardino, Faggian.

Retrieving some sequentiality: **Jumps**
 \swarrow seq \searrow

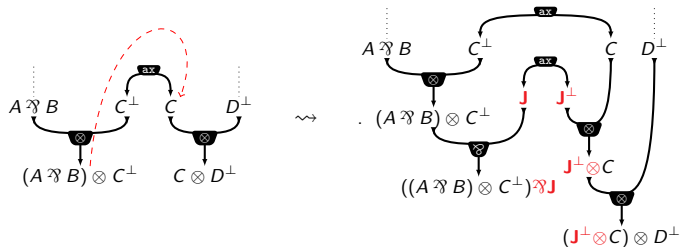
$$\frac{\frac{\frac{\overline{\vdash A^{\perp}, A} \quad \overline{\vdash B^{\perp}, B}}{\vdash A^{\perp} \otimes B^{\perp}, A, B} \otimes}{\vdash A^{\perp} \otimes B^{\perp}, A \wp B} \wp}{\vdash A^{\perp} \otimes B^{\perp}, (A \wp B) \otimes C^{\perp}, C} \otimes \quad \overline{\vdash C^{\perp}, C}}{\vdash A^{\perp} \otimes B^{\perp}, (A \wp B) \otimes C^{\perp}, C \otimes D^{\perp}, D} \otimes$$

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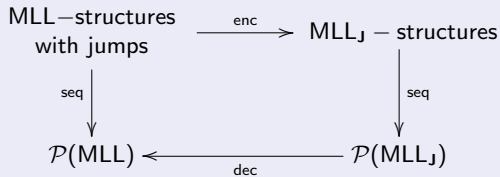
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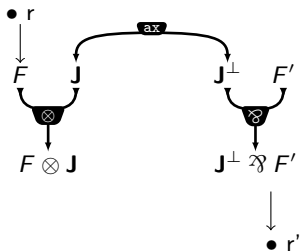
Roadmap: Encoding Jumps



Theorem



Key Idea: Encoding sequentiality



\rightarrow
seq

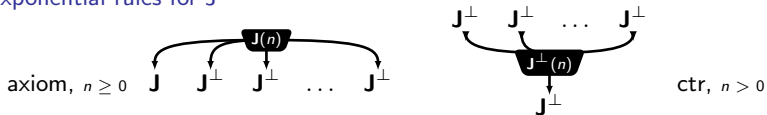
$$\begin{array}{c}
 \vdots \\
 \frac{\vdots}{\vdash \Gamma, F} \quad r \quad \frac{}{\vdash J, J^\perp} \quad ax \\
 \hline
 \vdash \Gamma, F \otimes J, J^\perp \quad \otimes \\
 \hline
 \vdots \\
 \frac{}{\vdash \Gamma', F', J^\perp} \quad \otimes \\
 \frac{}{\vdash \Gamma', F' \otimes J^\perp} \quad \otimes \\
 \hline
 r' \\
 \vdots
 \end{array}$$

\Rightarrow r appears **above** r' in any sequentialization

Multiple Jumps

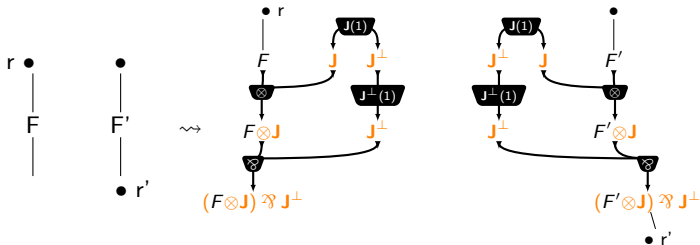
$$\text{MLL}_J \quad F_J = A \mid F_J \wp F_J \mid F_J \otimes F_J \mid F_J \wp J^\perp \mid F_J \otimes J$$

Exponential rules for J



Encoding steps:

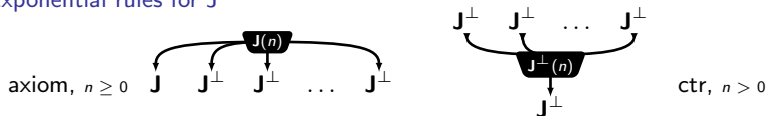
1) Replacing edges



Multiple Jumps

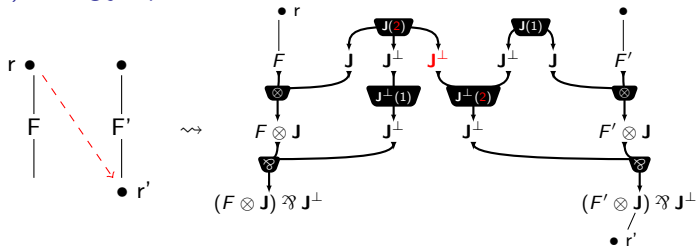
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Encoding steps:

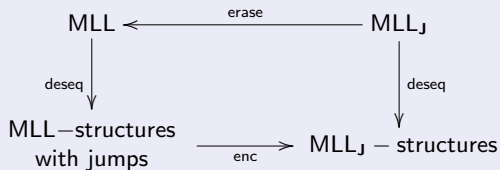
- 1) Replacing edges
- 2) Adding jumps



Multiple Jumps

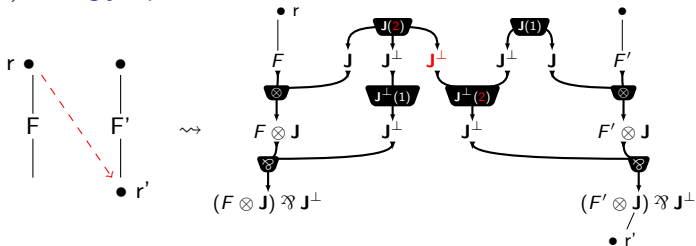
Theorem

Without cuts,



Encoding steps:

- 1) Replacing edges
- 2) Adding jumps



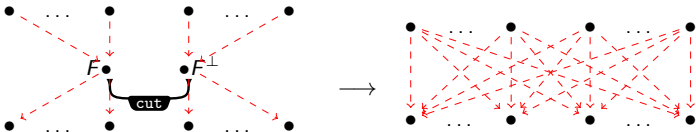
Dealing with cuts

Problems:

1 Retrieving symetries

$$\begin{array}{c}
 F \quad F^\perp \\
 \text{cut}
 \end{array}
 \quad \rightsquigarrow \quad
 (F \otimes J) \rightsquigarrow J^\perp (F^\perp \otimes J) \rightsquigarrow J^\perp$$

2 Propagating jumps



Dealing with cuts

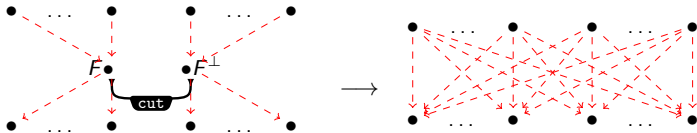
Problems:

1 Retrieving symetries

$$\begin{array}{c}
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 \quad \text{HH} \quad (F \otimes J) \cong J^\perp (F^\perp \otimes J) \cong J^\perp$$

⇒ **different** gadgets for positive and negative edges

2 Propagating jumps



Dealing with cuts

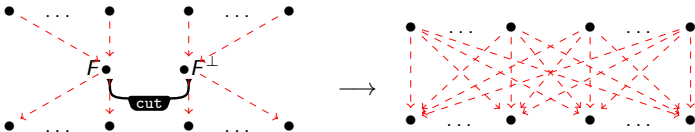
Problems:

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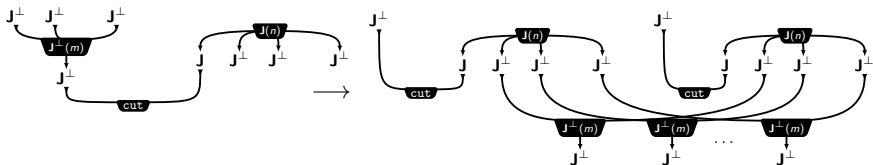
$$\begin{array}{c} F \quad F^\perp \\ \text{cut} \end{array} \quad \not\equiv \quad (F \otimes J) \cong J^\perp (F^\perp \otimes J) \cong J^\perp$$

\implies **different** gadgets for positive and negative edges

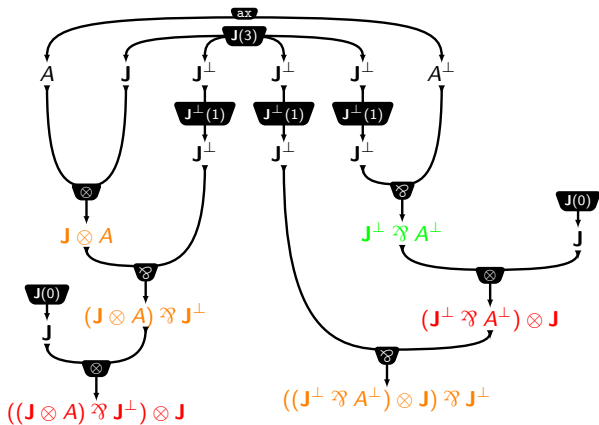
2 Propagating jumps



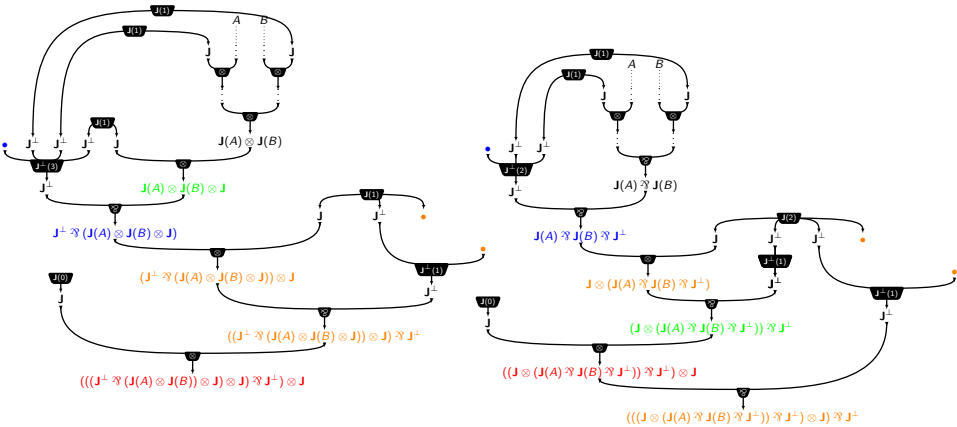
Exponential dynamics



Gadget for Axioms

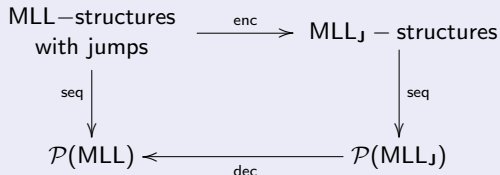


Gadget for Positive and Negative links



Correctness

Theorem

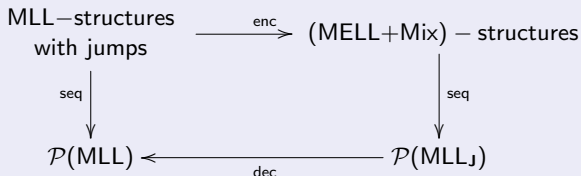


There exists a *bisimulation* procedure so that the diagram is stable under cut-elimination.

→ The encoding is **linear** in the number of links and jumps but cut elimination may induce an **exponential** growth of the number of jumps.

Correctness

Theorem



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→ The encoding is **linear** in the number of links and jumps but cut elimination may induce an **exponential** growth of the number of jumps.

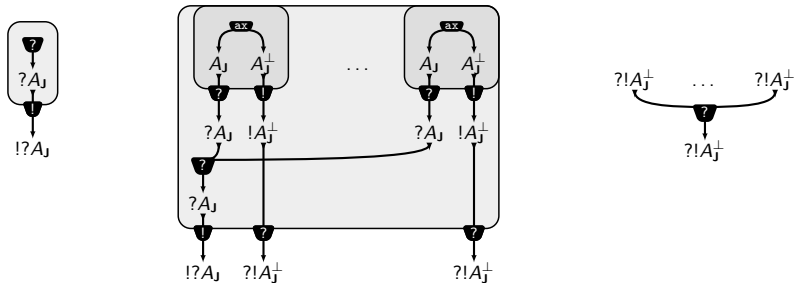
Encoding MLL_J in MELL + Mix

Encoding J

$$J = !?A_J$$

$$J^\perp = !?A_J^\perp$$

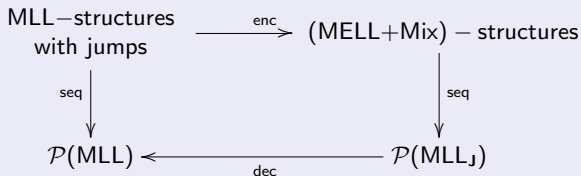
Encoding J rules



✓ Dynamics

The Exponential logic of sequentialization (Conclusion)

Theorem



There exists a *bisimulation* procedure so that the diagram is stable under cut-elimination.

Conjectures

- Easy extension of the encoding to other fragments of LL
- If $\text{dec}(\text{seq}(\text{enc}(R))) = \{\pi\}$ then $\text{dec}(\text{seq}(\text{enc}(\Downarrow R))) = \{\pi' \mid \pi \Downarrow \pi'\}$