Concurrent strategies for Herbrand's theorem

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CSL18 paper with

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PPS seminar, IRIF, Paris

Roadmap

Herbrand's witnesses

Herbrand's theorem (Simple)

A purely existential formula $\exists \bar{x}\ \varphi(\bar{x})$ *is valid in classical logic iff there is a finite set of witnesses* $\bar{t}_1, \ldots, \bar{t}_n \in \mathrm{Tm}_\Sigma$ *s.t.* $\models \varphi(\bar{t}_1) \vee \ldots \vee \varphi(\bar{t}_n)$*.*

Example $\models \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x}))$

$$\models (\neg D(c) \vee D(f(c))) \vee (\neg D(f(c)) \vee D(f(f(c))))$$

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Example $\models \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x}))$

$$\models (\neg D(c) \vee D(f(c))) \vee (\neg D(f(c)) \vee D(f(f(c))))$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ \neg D(c) \vee D(f(c)),\ \neg D(f(c)) \vee D(f(f(c)))}\ \textsc{Prop. Tautology}}{\vdash\ \neg D(c) \vee D(f(c)),\ \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x}))}\ \exists_I, \mathrm{x} := f(c)}{\vdash \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x})),\ \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x}))}\ \exists_I, \mathrm{x} := c}{\vdash \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x}))}\ \textsc{Contraction}$$

Herbrand proofs

Herbrand's theorem (General)

A 1^{st} order formula φ is valid in classical logic iff it has a Herbrand proof.

Example $\models \exists x \forall y,\ \neg D(x) \vee D(y)$ (DF)

A proof for DF:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash\ \neg D(c) \vee D(y),\ \neg D(y) \vee D(z))}\ \textsc{Prop. Tautology}}{\vdash\ \neg D(c) \vee D(y), \exists x \forall y,\ \neg D(x) \vee D(y)}\ \exists_I, x := y, \forall_I}{\vdash \exists x \forall y\ \neg D(x) \vee D(y), \exists x \forall y\ \neg D(x) \vee D(y)}\ \exists_I, x := c, \forall_I}{\vdash \exists x \forall y\ \neg D(x) \vee D(y)}\ \textsc{Contraction}$$

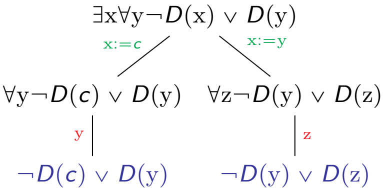
Herbrand proofs: Miller's **expansion trees**

Herbrand's theorem (Miller, 1987)

A 1^{st} order formula φ is valid in classical logic iff it has an expansion tree.

Example $\models \exists x \forall y,\ \neg D(x) \vee D(y)$ (DF)

An expansion tree for DF:



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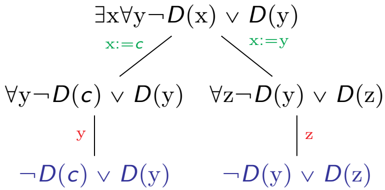

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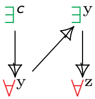


validity

$$\models (\neg D(c) \vee D(y)) \quad \vee \quad (\neg D(y) \vee D(z))$$

Herbrand proofs: Miller's **expansion trees**

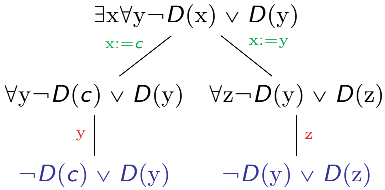
Herbrand's theorem (Miller, 1987)

A 1^{st} *order formula* φ *is valid in classical logic iff it has an expansion tree.*

Proof: By translation from the cut-free sequent calculus. → not compositional.

Example $\models \exists x \forall y,\ \neg D(x) \vee D(y)$ (DF)

An expansion tree for DF:



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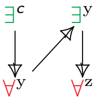


validity

$$\models (\neg D(c) \vee D(y)) \ \vee \ (\neg D(y) \vee D(z))$$

Toward compositionality?

Question: find a **composable** notion of expansion tree/Herbrand proof?

Syntactic approaches: Heijltjes, McKinley, Hetzl and Weller, via notions of Herbrand proofs with cuts.

Contribution (semantic approach): Expansion trees as strategies in a concurrent game model (categories of winning Σ-strategies).

Herbrand's theorem (Compositional Herbrand's theorem)

A 1^{st} *order formula* φ *is valid iff there is a winning Σ-strategy:* $\sigma : [\![\varphi]\!]$.

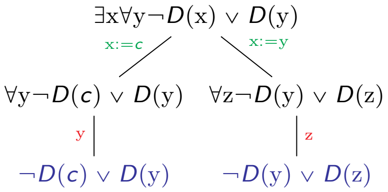
$$\frac{\vdash \Gamma, \varphi \qquad \vdash \Delta, \varphi^{\perp}}{\vdash \Gamma, \Delta}\ \textsc{Cut} \qquad\qquad \sigma = \sigma_1 \odot \sigma_2$$

Other related works: Games for first-order proofs (Laurent, Mimram)

Roadmap

From expansion trees to concurrent strategies

An implicit two-player game played on the formula between ∃loïse and ∀bélard:



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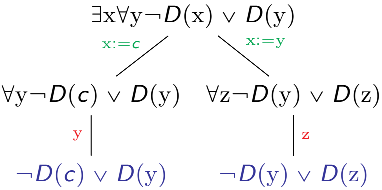
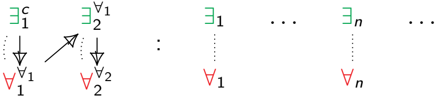


An interpretation of formulas as games and proofs as winning Σ-strategies:



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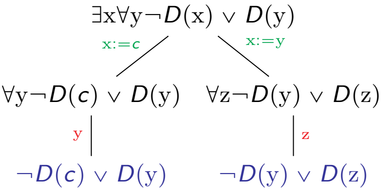

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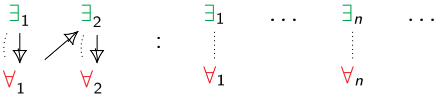

→ A causal game model

From expansion trees to concurrent strategies

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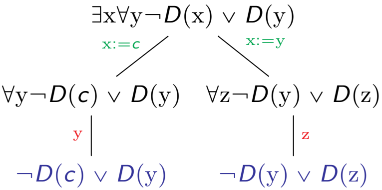

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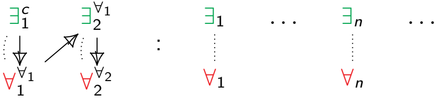
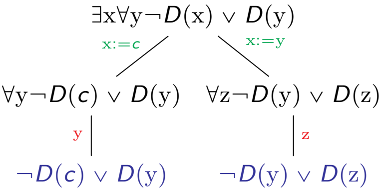


→ A causal game model with term labelling

From expansion trees to concurrent strategies

An implicit two-player game played on the formula between ∃loïse and ∀bélard:



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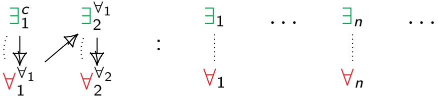


→ A causal game model with term labelling and winning conditions.

Concurrent arenas and strategies

Definition

A **arena** is a triple $(|A|, \leqslant_A, \mathrm{pol}_A)$, with:

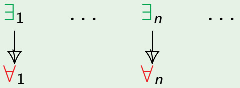
- $(|A|, \leqslant_A)$ a causal relation, i.e. a **partial order** with *finite* histories
- $\mathrm{pol}_A : |A| \to \{\forall, \exists\}$

Notation: $\mathcal{C}(A)$ is the set of configurations (down-closed subsets of A).

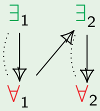
Definition

Strategies $\sigma : A$ are *certain* $(|\sigma|, \leqslant_\sigma)$, s.t. $\sigma \subseteq A$ and $\mathcal{C}(\sigma) \subseteq \mathcal{C}(A)$

The arena for $\exists \mathrm{x}\ \forall \mathrm{y}\ \psi(\mathrm{x}, \mathrm{y})$



A strategy on $\exists \mathrm{x}\ \forall \mathrm{y}\ \psi(\mathrm{x}, \mathrm{y})$



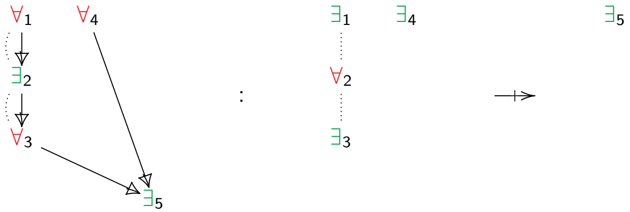
A (compact closed) category of arenas

Constructions on arenas.

- If A is an arena, $A^\perp$ has the same structure with polarity inverted.
- If A, B are arenas, $A \parallel B$ has events $|A| + |B|$, and components inherited.

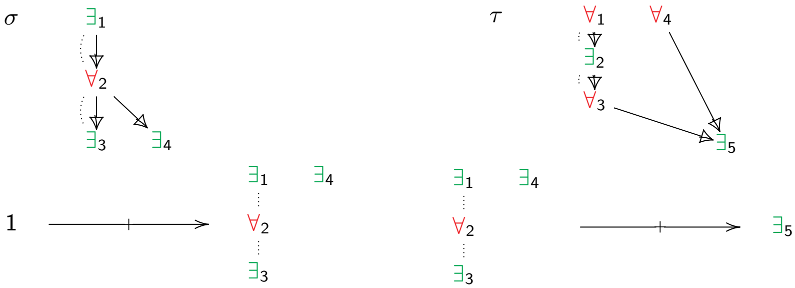
Definition

A **strategy from A to B** is $\sigma : A^\perp \parallel B$, written $\sigma : A \rightarrow\!\!\!\!+\!\!\!> B$.

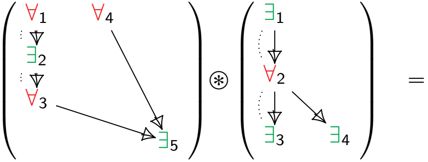


Composition $\tau \odot \sigma : A \rightarrow\!\!\!\!+\!\!\!> C$ is defined for all $\sigma : A \rightarrow\!\!\!\!+\!\!\!> B, \tau : B \rightarrow\!\!\!\!+\!\!\!> C$.

What is the result of the composition of the strategies σ and τ?



Interaction (a meet):



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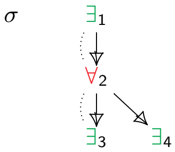

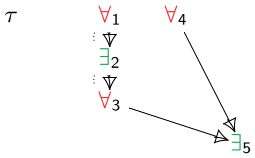

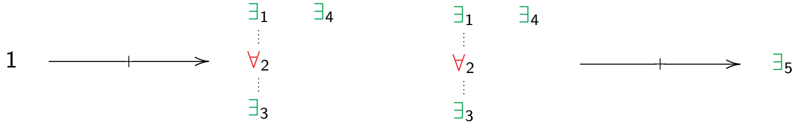

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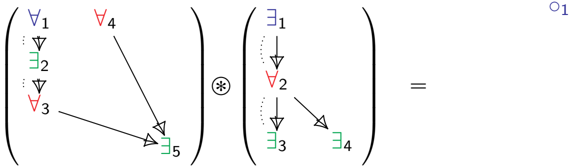

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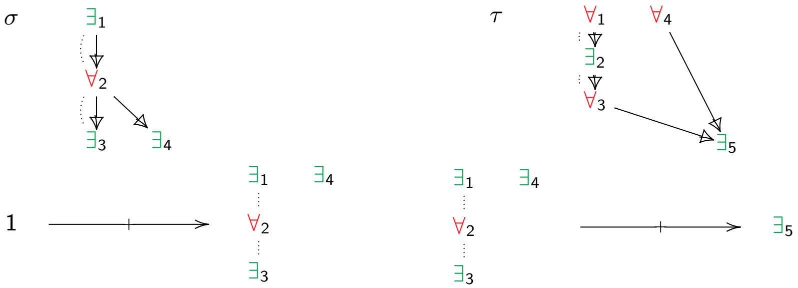

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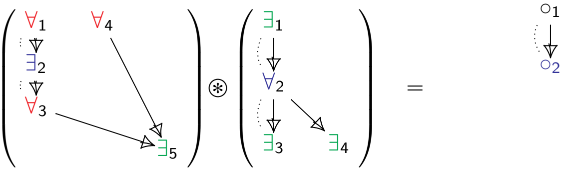

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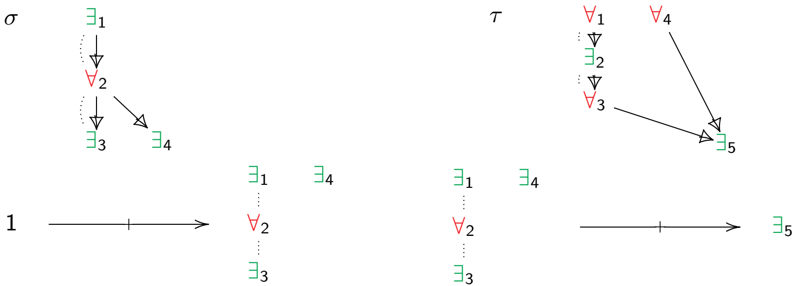

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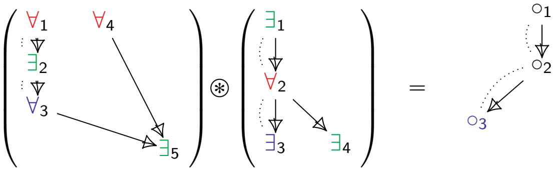

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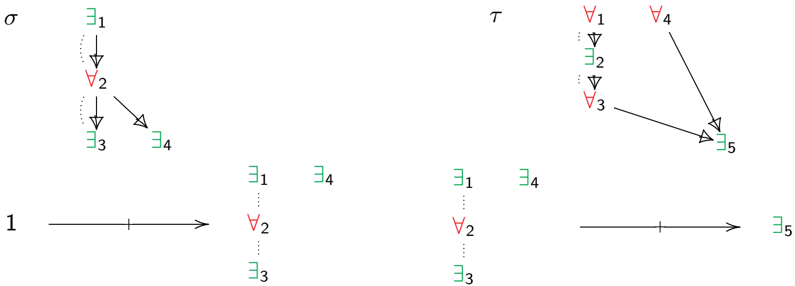

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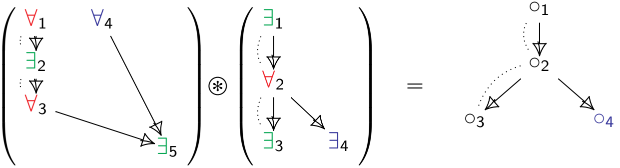

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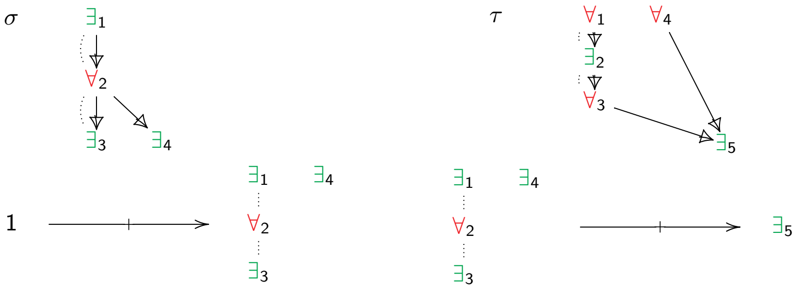

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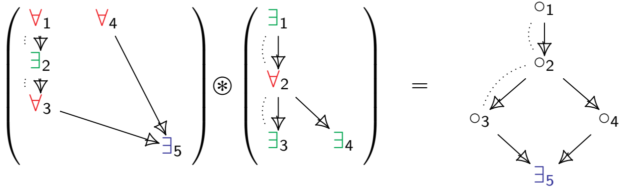

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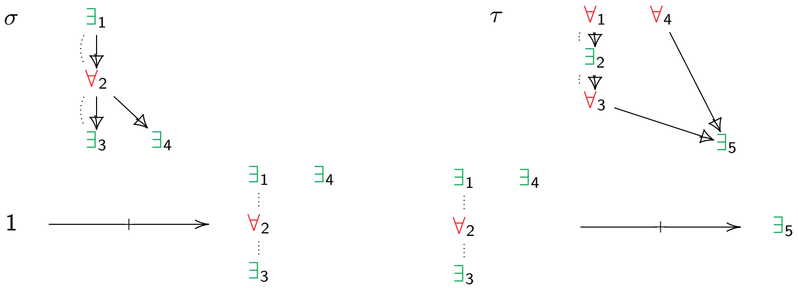

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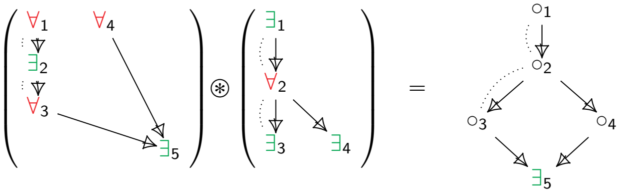

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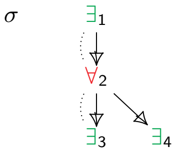
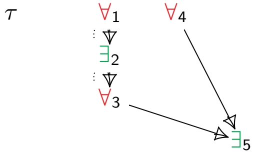
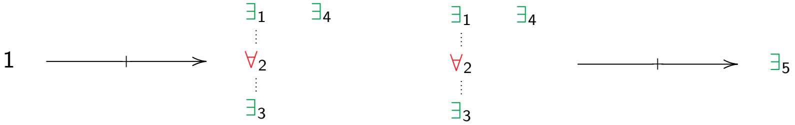
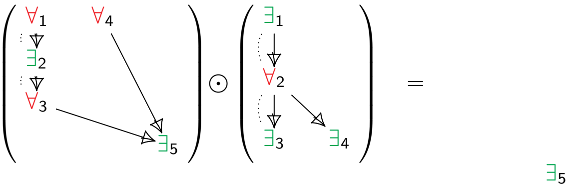






Composition (projection):



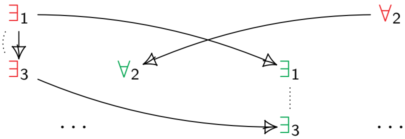
A (compact closed) category of arenas

Composition.

$$\tau \odot \sigma \quad = \quad \tau \circledast \sigma \downarrow A, C \quad : \quad A \,\mathrel{\rightarrow\!\!\!\!+}\, C$$

Identities. Copycat strategies:

$$\mathrm{id}_A \quad : \quad A \quad \longrightarrow\!\!\!\!\!\!\!\!+\quad A$$

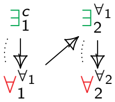


Compact closure.

$$\eta_A : \varnothing \,\mathrel{\rightarrow\!\!\!\!+}\, A^\perp \parallel A \qquad\qquad \epsilon_A : A \parallel A^\perp \,\mathrel{\rightarrow\!\!\!\!+}\, \varnothing$$

Σ-strategies on arenas

A **strategy**, plus **free variables** (∀bélard's moves) and **terms** (∃loïse's moves).



Definition

A **Σ-strategy** on A is a strategy $\sigma : A$, with a **labeling function**

$$\lambda_\sigma : |\sigma| \to \mathsf{Tm}_\Sigma(|\sigma|)$$

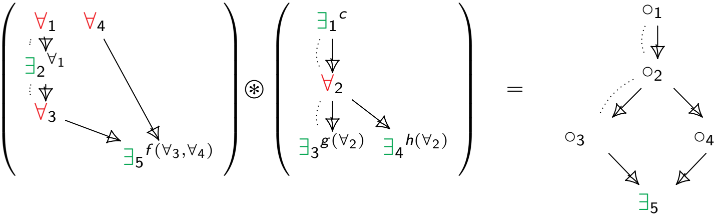
such that:

$$\begin{array}{lrcl} \forall a^{\forall} \in |\sigma|, & \lambda_\sigma(a) & = & a \\ \forall a^{\exists} \in |\sigma|, & \lambda_\sigma(a) & \in & \mathsf{Tm}_\Sigma([a]_\sigma^{\forall}) \end{array}$$

where $[a]_\sigma^{\forall} = \{a' \in |\sigma| \mid a' \leqslant_\sigma a \ \& \ \mathrm{pol}_A(a') = \forall\}$.

What is the result of the composition of the Σ-strategies σ and τ?

Same causal structure, with terms.



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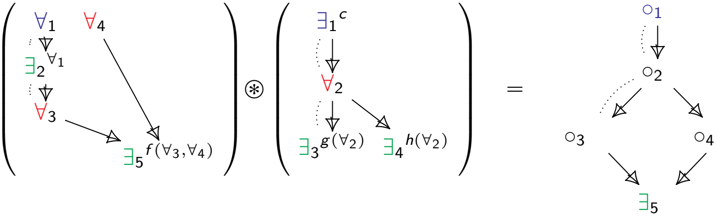

$\circ_1 \quad \doteq \quad c$

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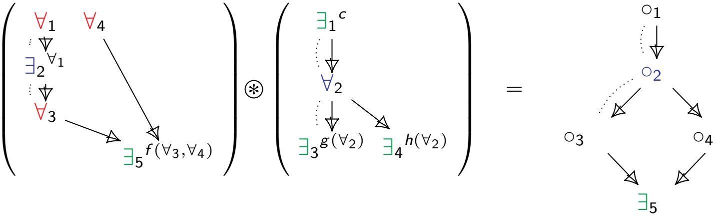

$$\circ_1 \doteq c$$
$$\circ_1 \doteq \circ_2$$

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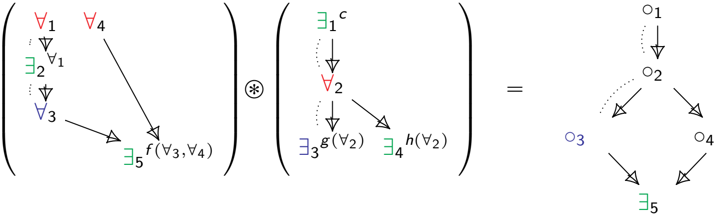

$$\circ_1 \doteq c$$
$$\circ_1 \doteq \circ_2$$
$$\circ_3 \doteq g(\circ_2)$$

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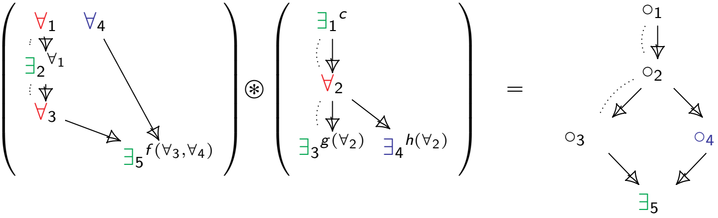

$$\circ_1 \doteq c$$
$$\circ_1 \doteq \circ_2$$
$$\circ_3 \doteq g(\circ_2)$$
$$\circ_4 \doteq h(\circ_2)$$

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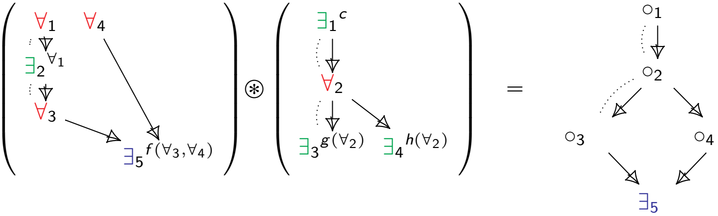

$$
\begin{aligned}
\circ_1 &\doteq c \\
\circ_1 &\doteq \circ_2 \\
\circ_3 &\doteq g(\circ_2) \\
\circ_4 &\doteq h(\circ_2) \\
f(\circ_3, \circ_4) &\doteq \exists_5
\end{aligned}
$$

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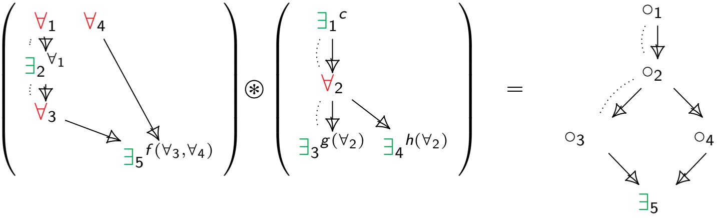

$$\left\{ \begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array} \right\}$$

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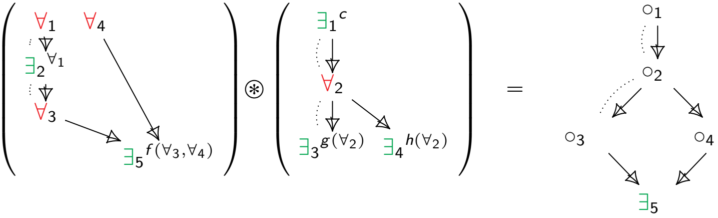

$$\left\{\begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array}\right\}$$

with **m.g.u.** $\left\{ \qquad \right\}$

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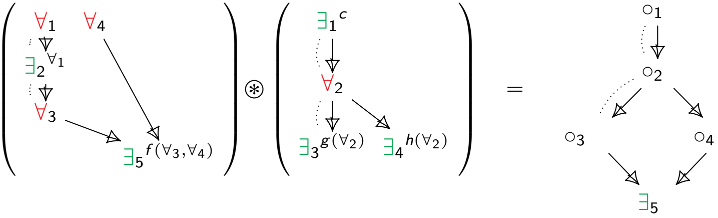

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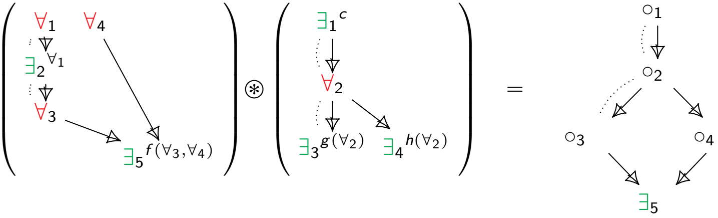

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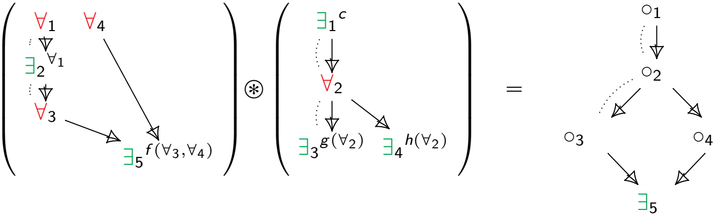

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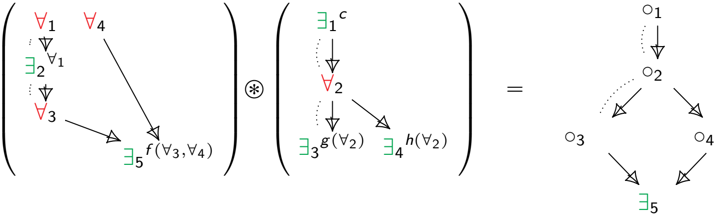

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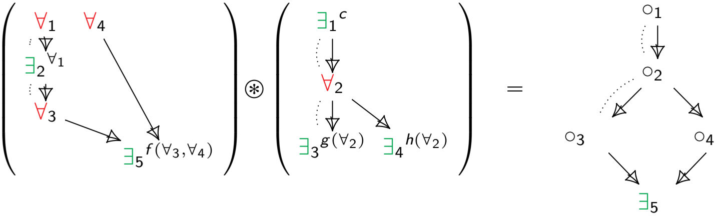

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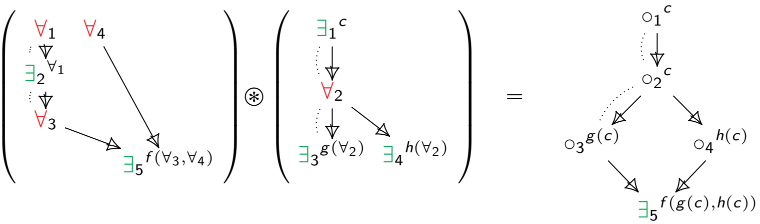

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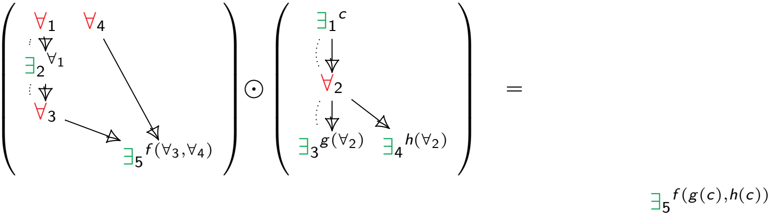

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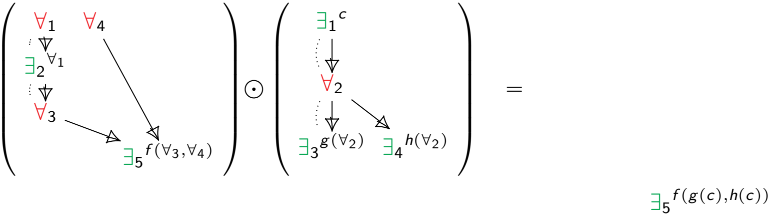

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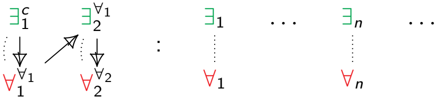
→ A **new** compact closed category.

Example of winning conditions

Consider the Σ-strategy $\sigma : [\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$ over DF



Validity in expansion trees:

$$\models (\neg D(c) \vee D(\forall_1)) \quad \vee \quad (\neg D(\forall_1) \vee D(\forall_2))$$

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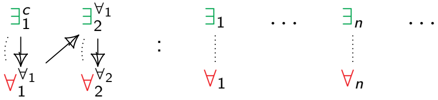

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Can be decomposed into

$$\models \quad \underbrace{(\neg D(\exists_1) \vee D(\forall_1)) \quad \vee \quad (\neg D(\exists_2) \vee D(\forall_2))}_{\textbf{Winning conditions},\ \mathcal{W}_{\mathrm{DF}}(|\sigma|)} \quad \underbrace{[\exists_1 \mapsto c; \exists_2 \mapsto \forall_1]}_{\text{Labelling},\ \lambda_\sigma}$$

Winning conditions on arenas

Definition

A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition. σ is a **winning on x** if $\models \mathcal{W}_A(x)[\lambda_\sigma]$.

To each configuration of $[\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$, we associate a **formula**:

$\exists_1$ $\exists_2$ $\dots$

$\forall_1$ $\forall_2$

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$\exists_1$

$\vdots$

$\forall_1$ $\quad\mapsto\quad$ $\neg D(\exists_1) \vee D(\forall_1)$

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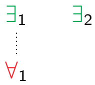

$$\mapsto \qquad (\neg D(\exists_1) \vee D(\forall_1)) \quad \vee \quad \top$$

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where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition

A Σ-strategy $\sigma : A$ is **winning on $\mathcal{W}_{\mathcal{A}}$** iff for all $x \in \mathcal{C}^{\infty}(\sigma)$ $\exists$-**maximal**,

$$\models \mathcal{W}_A(x)[\lambda_\sigma]$$

→ Two new constructors on games: $\otimes$ (conjunction) and $⅋$ (disjunction)
with units $1 = (\varnothing, \mathcal{W}_1(\varnothing) = \top)$ $\quad \perp = (\varnothing, \mathcal{W}_\perp(\varnothing) = \perp)$

→ Winning strategies $\sigma : \mathcal{A}^\perp ⅋ \mathcal{B}$ are **stable under composition ($\star$-autonomous category).**

Roadmap

Interpretation

Herbrand's theorem (Compositional Herbrand's theorem)

A 1^{st} order formula φ is valid iff there is a winning Σ-strategy: $\sigma : [\![\varphi]\!]$.

Proof:

$\Leftarrow$ Winning strategies ressemble expansion trees.

$\Rightarrow$ Interpret the classical sequent calculus LK.

$$\frac{\pi}{\vdash \varphi} \quad \rightsquigarrow \quad [\![\pi]\!] : [\![\varphi]\!]$$

LK sequent calculus

Identity group

$$\textsc{Ax}\ \frac{}{\vdash^{\mathcal{V}} \varphi, \varphi^{\perp}} \qquad \textsc{Cut}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \varphi^{\perp}, \Delta}{\vdash^{\mathcal{V}} \Gamma, \Delta}$$

Structural group

$$\mathrm{C}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \varphi}{\vdash^{\mathcal{V}} \Gamma, \varphi} \qquad \mathrm{W}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V}} \Gamma, \varphi} \qquad \textsc{Ex}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi, \Delta}{\vdash^{\mathcal{V}} \Gamma, \psi, \varphi, \Delta} \qquad \textsc{W-Var}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V} \uplus \{\mathrm{x}\}} \Gamma}$$

Propositional group

$$\perp\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V}} \Gamma, \perp} \qquad \top\mathrm{I}\ \frac{}{\vdash^{\mathcal{V}} \top} \qquad \wedge\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \Delta, \psi}{\vdash^{\mathcal{V}} \Gamma, \Delta, \varphi \wedge \psi} \qquad \vee\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi}{\vdash^{\mathcal{V}} \Gamma, \varphi \vee \psi}$$

Quantifiers group

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V} \uplus \{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma) \qquad \exists\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathrm{x}]}{\vdash^{\mathcal{V}} \Gamma, \exists \mathrm{x}.\, \varphi}\ t \in \mathsf{Tm}_{\Sigma}(\mathcal{V})$$

MLL sequent calculus

Identity group

$$\textsc{Ax}\ \frac{}{\vdash\ \varphi, \varphi^{\perp}} \qquad \textsc{Cut}\ \frac{\vdash\ \Gamma, \varphi \qquad \vdash\ \varphi^{\perp}, \Delta}{\vdash\ \Gamma, \Delta}$$

Structural group

$$\textsc{Ex}\ \frac{\vdash\ \Gamma, \varphi, \psi, \Delta}{\vdash\ \Gamma, \psi, \varphi, \Delta}$$

Propositional group

$$\perp\mathrm{I}\ \frac{\vdash\ \Gamma}{\vdash\ \Gamma, \perp} \qquad \top\mathrm{I}\ \frac{}{\vdash\ \top} \qquad \wedge\mathrm{I}\ \frac{\vdash\ \Gamma, \varphi \qquad \vdash\ \Delta, \psi}{\vdash\ \Gamma, \Delta, \varphi \wedge \psi} \qquad \vee\mathrm{I}\ \frac{\vdash\ \Gamma, \varphi, \psi}{\vdash\ \Gamma, \varphi \vee \psi}$$

a $\star$-autonomous category Games.

MLL ⋆-autonomous model

Proof as morphisms:

$$\frac{\pi}{\vdash \varphi_1, \dots, \varphi_n} \quad \rightsquigarrow \quad [\![\pi]\!] : \mathbf{1} \longrightarrow [\![\varphi_1]\!] \parr \cdots \parr [\![\varphi_n]\!]$$

Propositional connectives.

$$[\![\top]\!] = \mathbf{1} \qquad [\![\bot]\!] = \bot$$

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \otimes [\![\psi]\!] \qquad [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \parr [\![\psi]\!]$$

Rules.

$$\text{Ax}\ \frac{}{\vdash \varphi, \varphi^\perp} \quad \rightsquigarrow \quad \mathsf{id}_A$$

$$\frac{\vdash \Gamma, \varphi \qquad \vdash \Delta, \varphi^\perp}{\vdash \Gamma, \Delta}\ \text{Cut} \quad \rightsquigarrow \quad [\![\pi_1]\!] \odot [\![\pi_2]\!]$$

...

MLL sequent calculus

Identity group

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Structural group

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a $\star$-autonomous category Games.

$\mathbf{MLL_1}$ sequent calculus

Identity group

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Structural group

$$\textsc{Ex}\ \frac{\vdash\ \Gamma, \varphi, \psi, \Delta}{\vdash\ \Gamma, \psi, \varphi, \Delta}$$

Propositional group

$$\perp\mathrm{I}\ \frac{\vdash\ \Gamma}{\vdash\ \Gamma, \perp} \qquad \top\mathrm{I}\ \frac{}{\vdash\ \top} \qquad \wedge\mathrm{I}\ \frac{\vdash\ \Gamma, \varphi \qquad \vdash\ \Delta, \psi}{\vdash\ \Gamma, \Delta, \varphi \wedge \psi} \qquad \vee\mathrm{I}\ \frac{\vdash\ \Gamma, \varphi, \psi}{\vdash\ \Gamma, \varphi \vee \psi}$$

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a $\star$-autonomous category Games.

MLL$_1$ sequent calculus

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$$\text{Ax}\ \frac{}{\vdash^{\mathcal{V}} \varphi, \varphi^{\perp}} \qquad \text{Cut}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \varphi^{\perp}, \Delta}{\vdash^{\mathcal{V}} \Gamma, \Delta}$$

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Propositional group

$$\bot\text{I}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V}} \Gamma, \bot} \qquad \top\text{I}\ \frac{}{\vdash^{\mathcal{V}} \top} \qquad \wedge\text{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \Delta, \psi}{\vdash^{\mathcal{V}} \Gamma, \Delta, \varphi \wedge \psi} \qquad \vee\text{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi}{\vdash^{\mathcal{V}} \Gamma, \varphi \vee \psi}$$

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For each $\mathcal{V}$, a $\star$-autonomous category $\mathcal{V}$-Games. $\exists x/\forall x$ as functors

MLL_1 model

Quantifiers.

$$[\![\exists x \varphi]\!]_{\mathcal{V}} = \exists_{x}.[\![\varphi]\!]_{\mathcal{V} \uplus \{x\}} \qquad [\![\forall x \varphi]\!]_{\mathcal{V}} = \forall_{x}.[\![\varphi]\!]_{\mathcal{V} \uplus \{x\}}$$

$$\exists.A = \begin{array}{c} \exists \\ \downarrow \\ A \end{array}, \quad \begin{array}{l} \mathcal{W}_{\exists.\mathcal{A}}(\emptyset) = \bot \\ \mathcal{W}_{\exists.\mathcal{A}}(\exists.x_A) = \mathcal{W}_{\mathcal{A}}(x_{\mathcal{A}}) \end{array}$$

$$\forall.A = \begin{array}{c} \forall \\ \downarrow \\ A \end{array}, \quad \begin{array}{l} \mathcal{W}_{\forall.\mathcal{A}}(\emptyset) = \top \\ \mathcal{W}_{\forall.\mathcal{A}}(\forall.x_A) = \mathcal{W}_{\mathcal{A}}(x_{\mathcal{A}}) \end{array}$$

MLL_1 model

Quantifiers.

$$[\![\exists x\varphi]\!]_{\mathcal{V}} = \exists_{x}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}} \qquad [\![\forall x\varphi]\!]_{\mathcal{V}} = \forall_{x}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}}$$

$\exists.A =$

$$\begin{array}{c} \exists \\ \downarrow \\ A \end{array}, \qquad \begin{array}{l} \mathcal{W}_{\exists.\mathcal{A}}(\emptyset) = \bot \\ \mathcal{W}_{\exists.\mathcal{A}}(\exists.x_A) = \mathcal{W}_{\mathcal{A}}(x_{\mathcal{A}}) \end{array}$$

$\forall.A =$

$$\begin{array}{c} \forall \\ \downarrow \\ A \end{array}, \qquad \begin{array}{l} \mathcal{W}_{\forall.\mathcal{A}}(\emptyset) = \top \\ \mathcal{W}_{\forall.\mathcal{A}}(\forall.x_A) = \mathcal{W}_{\mathcal{A}}(x_{\mathcal{A}}) \end{array}$$

Rules.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{x\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall x.\, \varphi}\ x \notin \mathcal{V}, x \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}\uplus\{x\}} \parr \qquad [\![\varphi]\!]_{\mathcal{V}\uplus\{x\}}$$

MLL_1 model

Quantifiers.

$$[\![\exists x \varphi]\!]_{\mathcal{V}} = \exists_x.[\![\varphi]\!]_{\mathcal{V} \uplus \{x\}} \qquad [\![\forall x \varphi]\!]_{\mathcal{V}} = \forall_x.[\![\varphi]\!]_{\mathcal{V} \uplus \{x\}}$$

$$\exists.A = \begin{array}{c} \exists \\ \downarrow \\ A \end{array}, \quad \begin{array}{l} \mathcal{W}_{\exists.\mathcal{A}}(\varnothing) = \bot \\ \mathcal{W}_{\exists.\mathcal{A}}(\exists.x_A) = \mathcal{W}_{\mathcal{A}}(x_{\mathcal{A}}) \end{array}$$

$$\forall.A = \begin{array}{c} \forall \\ \downarrow \\ A \end{array}, \quad \begin{array}{l} \mathcal{W}_{\forall.\mathcal{A}}(\varnothing) = \top \\ \mathcal{W}_{\forall.\mathcal{A}}(\forall.x_A) = \mathcal{W}_{\mathcal{A}}(x_{\mathcal{A}}) \end{array}$$

Rules.

$$\forall I \; \frac{\vdash^{\mathcal{V} \uplus \{x\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall x.\, \varphi} \; x \notin \mathcal{V}, x \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}} \qquad ⅋ \; \forall x.\; [\![\varphi]\!]_{\mathcal{V} \uplus \{x\}}$$

MLL_1 model

Quantifiers.

$$[\![\exists x \varphi]\!]_{\mathcal{V}} = \exists_x . [\![\varphi]\!]_{\mathcal{V} \uplus \{x\}} \qquad [\![\forall x \varphi]\!]_{\mathcal{V}} = \forall_x . [\![\varphi]\!]_{\mathcal{V} \uplus \{x\}}$$

$$\exists . A = \begin{array}{c} \exists \\ \downarrow \\ A \end{array} , \quad \begin{array}{l} \mathcal{W}_{\exists.\mathcal{A}}(\varnothing) = \bot \\ \mathcal{W}_{\exists.\mathcal{A}}(\exists . x_{\mathcal{A}}) = \mathcal{W}_{\mathcal{A}}(x_{\mathcal{A}}) \end{array}$$

$$\forall . A = \begin{array}{c} \forall \\ \downarrow \\ A \end{array} , \quad \begin{array}{l} \mathcal{W}_{\forall.\mathcal{A}}(\varnothing) = \top \\ \mathcal{W}_{\forall.\mathcal{A}}(\forall . x_{\mathcal{A}}) = \mathcal{W}_{\mathcal{A}}(x_{\mathcal{A}}) \end{array}$$

Rules.

$$\exists \mathrm{I} \; \frac{\vdash^{\mathcal{V}} \Gamma, \varphi[t/x]}{\vdash^{\mathcal{V}} \Gamma, \exists x . \varphi} \; t \in \mathsf{Tm}_{\Sigma}(\mathcal{V})$$

Composition with

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}} \parr [\![\varphi[t/x]]\!]_{\mathcal{V}} \qquad u_{\mathcal{A},t} : \mathcal{A}[t/x] \xrightarrow{\mathcal{V}\text{-Games}} \exists x \mathcal{A}$$

$\mathbf{MLL_1}$ sequent calculus

Identity group

$$\textsc{Ax}\ \frac{}{\vdash^{\mathcal{V}} \varphi, \varphi^{\perp}} \qquad \textsc{Cut}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \varphi^{\perp}, \Delta}{\vdash^{\mathcal{V}} \Gamma, \Delta}$$

Structural group

$$\textsc{Ex}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi, \Delta}{\vdash^{\mathcal{V}} \Gamma, \psi, \varphi, \Delta} \qquad \textsc{W-Var}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V} \uplus \{x\}} \Gamma}$$

Propositional group

$$\perp\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V}} \Gamma, \perp} \qquad \top\mathrm{I}\ \frac{}{\vdash^{\mathcal{V}} \top} \qquad \wedge\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \Delta, \psi}{\vdash^{\mathcal{V}} \Gamma, \Delta, \varphi \wedge \psi} \qquad \vee\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi}{\vdash^{\mathcal{V}} \Gamma, \varphi \vee \psi}$$

Quantifiers group

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V} \uplus \{x\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall x.\, \varphi}\ x \notin \mathcal{V}, x \notin \mathrm{fv}(\Gamma) \qquad \exists\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi[t/x]}{\vdash^{\mathcal{V}} \Gamma, \exists x.\, \varphi}\ t \in \mathrm{Tm}_{\Sigma}(\mathcal{V})$$

For each $\mathcal{V}$, a $\star$-autonomous category $\mathcal{V}$-Games. $\exists x / \forall x$ as functors

LK sequent calculus

Identity group

$$\textsc{Ax}\ \frac{}{\vdash^{\mathcal{V}} \varphi, \varphi^{\perp}} \qquad \textsc{Cut}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \varphi^{\perp}, \Delta}{\vdash^{\mathcal{V}} \Gamma, \Delta}$$

Structural group

$$\mathrm{C}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \varphi}{\vdash^{\mathcal{V}} \Gamma, \varphi} \qquad \mathrm{W}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V}} \Gamma, \varphi} \qquad \textsc{Ex}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi, \Delta}{\vdash^{\mathcal{V}} \Gamma, \psi, \varphi, \Delta} \qquad \textsc{W-Var}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V} \uplus \{\mathrm{x}\}} \Gamma}$$

Propositional group

$$\perp\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma}{\vdash^{\mathcal{V}} \Gamma, \perp} \qquad \top\mathrm{I}\ \frac{}{\vdash^{\mathcal{V}} \top} \qquad \wedge\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi \qquad \vdash^{\mathcal{V}} \Delta, \psi}{\vdash^{\mathcal{V}} \Gamma, \Delta, \varphi \wedge \psi} \qquad \vee\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi, \psi}{\vdash^{\mathcal{V}} \Gamma, \varphi \vee \psi}$$

Quantifiers group

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V} \uplus \{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma) \qquad \exists\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathrm{x}]}{\vdash^{\mathcal{V}} \Gamma, \exists \mathrm{x}.\, \varphi}\ t \in \mathsf{Tm}_{\Sigma}(\mathcal{V})$$

For each $\mathcal{V}$, a $\star$-autonomous category $\mathcal{V}$-Games. $\exists x / \forall x$ as functors

Exponentials constructors

LK model

Propositional connectives (MLL $\star$-autonomous model)

Quantifiers

$$[\![\exists \mathrm{x}\varphi]\!]_{\mathcal{V}} = \quad \exists_{\mathrm{x}}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}} \qquad\qquad [\![\forall \mathrm{x}\varphi]\!]_{\mathcal{V}} = \quad \forall_{\mathrm{x}}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}}$$

$\longrightarrow$ A model for proofs of first order MLL

LK model

Propositional connectives (MLL ⋆-autonomous model)

Quantifiers and exponentials

$$[\![\exists x \varphi]\!]_{\mathcal{V}} = ?\ \exists_x.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}} \qquad [\![\forall x \varphi]\!]_{\mathcal{V}} = \ \forall_x.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}}$$

$$?\ A = \|_{n\in\omega} A$$

$$\mathcal{W}_{?\mathcal{A}}(\|_i\ x_i) = \bigvee_i \mathcal{W}_{\mathcal{A}}(x_i)$$

→ A model for proofs of first order MLL

LK model

Propositional connectives (MLL $\star$-autonomous model)

Quantifiers and exponentials

$$[\![\exists x \varphi]\!]_{\mathcal{V}} = ?\ \exists_{x}.[\![\varphi]\!]_{\mathcal{V} \uplus \{x\}} \qquad [\![\forall x \varphi]\!]_{\mathcal{V}} = !\ \forall_{x}.[\![\varphi]\!]_{\mathcal{V} \uplus \{x\}}$$

$$?\ A = \|_{n \in \omega} A \qquad !\ A = \|_{n \in \omega} A$$

$$\mathcal{W}_{?\mathcal{A}}(\|_i\ x_i) = \bigvee_i \mathcal{W}_{\mathcal{A}}(x_i) \qquad \mathcal{W}_{!\mathcal{A}}(\|_i\ x_i) = \bigwedge_i \mathcal{W}_{\mathcal{A}}(x_i)$$

$\rightarrow$ A model for proofs of first order MLL

LK model

Propositional connectives (MLL $\star$-autonomous model)

Quantifiers and exponentials

$$[\![\exists \mathrm{x}\varphi]\!]_{\mathcal{V}} = ?\ \exists_{\mathrm{x}}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}} \qquad [\![\forall \mathrm{x}\varphi]\!]_{\mathcal{V}} = !\ \forall_{\mathrm{x}}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}}$$

$$?\ A = \|_{n\in\omega} A \qquad !\ A = \|_{n\in\omega} A$$

$$\mathcal{W}_{?\mathcal{A}}(\|_i\ x_i) = \bigvee_i \mathcal{W}_{\mathcal{A}}(x_i) \qquad \mathcal{W}_{!\mathcal{A}}(\|_i\ x_i) = \bigwedge_i \mathcal{W}_{\mathcal{A}}(x_i)$$

Contraction : for any formula φ, $c_{[\![\varphi]\!]} : [\![\varphi]\!] \twoheadrightarrow [\![\varphi]\!] \otimes [\![\varphi]\!]$

$$c_{[\![\varphi]\!]} = [\![\varphi]\!] \twoheadrightarrow ![\![\varphi]\!] \twoheadrightarrow ![\![\varphi]\!] \otimes ![\![\varphi]\!] \twoheadrightarrow [\![\varphi]\!] \otimes [\![\varphi]\!]$$

$\rightarrow$ A model for proofs of first order MLL + contraction (first order LK)

Interpreting the proof of the drinker's formula

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y})}{\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})}}{\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}$$

$$(\neg D(c), \; D(\mathrm{y}) \;, \neg D(\mathrm{y}) \;, \forall \mathrm{y} D(\mathrm{y}))$$

$\forall_2$

Interpreting the proof of the drinker's formula

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{y} \neg D(c), D(y), \neg D(y), \forall y D(y)}{\vdash^{y} \neg D(c), D(y), D(y) \Rightarrow \forall y D(y)}}{\vdash^{y} \neg D(c), D(y), \exists x (D(x) \Rightarrow \forall y D(y))}}{\vdash \neg D(c), \forall y D(y), \exists x (D(x) \Rightarrow \forall y D(y))}}{\vdash D(c) \Rightarrow \forall y D(y), \exists x (D(x) \Rightarrow \forall y D(y))}}{\vdash \exists x (D(x) \Rightarrow \forall y D(y)), \exists x (D(x) \Rightarrow \forall y D(y))}}{\vdash \exists x (D(x) \Rightarrow \forall y D(y))}$$

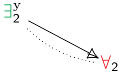
$$(\neg D(c),\ D(y),\ (D(y) \Rightarrow \forall y D(y)))$$

$\forall_2$

Interpreting the proof of the drinker's formula

$$
\begin{array}{c}
\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y}) \\ \hline
\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}) \\ \hline
\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\ \hline
\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\ \hline
\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\ \hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\ \hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))
\end{array}
$$

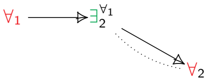
$$(\neg D(c) \,,\; D(\mathrm{y}) \;, \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})))$$



Interpreting the proof of the drinker's formula

$$
\begin{array}{c}
\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y}) \\
\hline
\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}) \\
\hline
\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))
\end{array}
$$

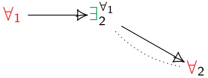
$\neg D(c)\ \ ,\ \forall \mathrm{y} D(\mathrm{y})\ \ ,\ \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$



Interpreting the proof of the drinker's formula

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y})}{\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})}}{\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}$$

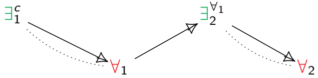
$$D(c) \Rightarrow \quad \forall \mathrm{y} D(\mathrm{y}) \ , \ \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$



Interpreting the proof of the drinker's formula

$$\begin{array}{c}
\vdash^{y} \neg D(c), D(y), \neg D(y), \forall y D(y) \\ \hline
\vdash^{y} \neg D(c), D(y), D(y) \Rightarrow \forall y D(y) \\ \hline
\vdash^{y} \neg D(c), D(y), \exists x(D(x) \Rightarrow \forall y D(y)) \\ \hline
\vdash \neg D(c), \forall y D(y), \exists x(D(x) \Rightarrow \forall y D(y)) \\ \hline
\vdash D(c) \Rightarrow \forall y D(y), \exists x(D(x) \Rightarrow \forall y D(y)) \\ \hline
\vdash \exists x(D(x) \Rightarrow \forall y D(y)), \exists x(D(x) \Rightarrow \forall y D(y)) \\ \hline
\vdash \exists x(D(x) \Rightarrow \forall y D(y))
\end{array}$$

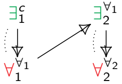
$$\exists x(D(x) \Rightarrow \quad \forall y D(y)) \ , \ \exists x(D(x) \Rightarrow \forall y D(y))$$



Interpreting the proof of the drinker's formula

$$
\begin{array}{c}
\vdash^{y} \neg D(c), D(y), \neg D(y), \forall y D(y) \\
\hline
\vdash^{y} \neg D(c), D(y), D(y) \Rightarrow \forall y D(y) \\
\hline
\vdash^{y} \neg D(c), D(y), \exists x(D(x) \Rightarrow \forall y D(y)) \\
\hline
\vdash \neg D(c), \forall y D(y), \exists x(D(x) \Rightarrow \forall y D(y)) \\
\hline
\vdash D(c) \Rightarrow \forall y D(y), \exists x(D(x) \Rightarrow \forall y D(y)) \\
\hline
\vdash \exists x(D(x) \Rightarrow \forall y D(y)), \exists x(D(x) \Rightarrow \forall y D(y)) \\
\hline
\vdash \exists x(D(x) \Rightarrow \forall y D(y))
\end{array}
$$

$$\exists x(D(x) \Rightarrow \forall y D(y))$$



Conclusion

Herbrand's theorem (Compositional Herbrand's theorem)

A 1^{st} order formula φ is valid iff there is a winning Σ-strategy: $\sigma : [\![\varphi]\!]$.

On cut free proofs $\sim$ Expansion trees with explicit acyclicity witness

Games$_\Sigma$: a concurrent game model with **terms** and **winnings**.

Proof:
a framework to interpret **first order proofs** (using extra constructors $\forall, \exists, !, ?$)

- $\longrightarrow$ Does not preserve cuts elimination in LK (by necessity)
- $\longrightarrow$ Reflects some dynamics of LK: infinite strategies.

Future investigation: a finitary composition of strategies?

Back to Herbrand proofs

Herbrand witnesses

$$\sigma \quad : \quad [\![\exists x \varphi(x)]\!]$$

$$\exists_1^{t_1} \quad \ldots \quad \exists_k^{t_k} \quad : \quad \exists_1 \quad \ldots \quad \exists_k \quad \ldots$$

$$\models \bigvee_{i=1}^{k} \varphi(t_i)$$

Back to Herbrand proofs

Herbrand witnesses

$$\sigma \quad : \quad [\![\exists x \varphi(x)]\!]$$

$$\exists_1^{t_1} \quad \ldots \quad \exists_k^{t_k} \quad \ldots \quad : \quad \exists_1 \quad \ldots \quad \exists_k \quad \ldots$$

$$\models \bigvee_{i=1}^{\infty} \varphi(t_i)$$

Proposition

There exists $\pi \vdash \exists x \varphi(x)$ *s.t.* $[\![\pi]\!] : [\![\exists x \varphi(x)]\!]$ *is infinite.*

Back to Herbrand proofs

Herbrand witnesses

$$\sigma \quad : \quad [\![\exists x \varphi(x)]\!]$$

$$\exists_1^{t_1} \quad \ldots \quad \exists_k^{t_k} \quad \ldots \quad : \quad \exists_1 \quad \ldots \quad \exists_k \quad \ldots$$

$$\models \bigvee_{i=1}^{\infty} \varphi(t_i) \quad \rightsquigarrow \quad \models \bigvee_{i=1}^{k} \varphi(t_i)$$

Proposition

There exists $\pi \vdash \exists x \varphi(x)$ s.t. $[\![\pi]\!] : [\![\exists x \varphi(x)]\!]$ is infinite.

Proposition (Compactness)

From every winning strategy $\sigma : [\![\exists x.\varphi(x)]\!]$ one can **effectively** *derived a finite winning sub-strategy $\sigma' : [\![\exists x.\varphi(x)]\!]$.*

(Can be generalised to all formulas)

Interpreting ∀I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}} \parr \qquad [\![\varphi]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}}$$

Interpreting $\forall$I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}} \parr \forall \mathrm{x}.\ [\![\varphi]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}}$$

Interpreting ∀I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V} \uplus \{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}} \qquad \invamp\ \forall \mathrm{x}.\ [\![\varphi]\!]_{\mathcal{V} \uplus \{\mathrm{x}\}}$$

Interpreting ∀I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}} \qquad ⅋\ \forall \mathrm{x}.\ [\![\varphi]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}}$$

All $\exists^t$ moves where $\mathrm{x} \in \mathrm{fv}(t)$ are set to depend on $\forall$.

Interpreting $\forall$I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}} \qquad \mathbin{⅋}\ \forall \mathrm{x}.\ [\![\varphi]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}}$$

All $\exists^{t}$ moves where $\mathrm{x} \in \mathrm{fv}(t)$ are set to depend on $\forall$.

The variable x is replaced by $\forall$ in λ_σ.

Interpreting ∃I

$$\exists\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathrm{x}]}{\vdash^{\mathcal{V}} \Gamma, \exists \mathrm{x}.\, \varphi}\ t \in \mathsf{Tm}_{\Sigma}(\mathcal{V})$$

Composition with the winning Σ-strategy

$$u_{\mathcal{A},t} : \mathcal{A}[t/\mathrm{x}] \overset{\mathcal{V}\text{-Games}}{\longrightarrow\!\!\!\!\!+\!\!\!>} \exists \mathrm{x}\mathcal{A}$$

playing $\exists^t$, then copycat on A.

Composition of plain strategies

Interaction. An **elementary event structure** is a partial order $(|\mathbf{q}|, \leqslant_{\mathbf{q}})$ such that for any $e \in |\mathbf{q}|$, $[e]_{\mathbf{q}}$ is finite.

Proposition

For $\mathbf{q}, \mathbf{q}'$, *we say that*

$$\mathbf{q} \preccurlyeq \mathbf{q}' \quad \Leftrightarrow \quad |\mathbf{q}| \subseteq |\mathbf{q}'| \ \&\ \mathcal{C}^\infty(\mathbf{q}) \subseteq \mathcal{C}^\infty(\mathbf{q}')$$

Then any two $\mathbf{q}$, $\mathbf{q}'$ *have a greatest lower bound (meet-semilattice).*

For $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$, define (ignoring polarities)

$$\tau \circledast \sigma = (\sigma \parallel C) \wedge (A \parallel \tau)$$

Composition. Define

$$\tau \odot \sigma \quad = \quad \tau \circledast \sigma \downarrow A, C \quad : \quad A \rightarrow\!\!\!\!\!+\!\!\!\!> C$$

Composition of winning strategies

Constructions. If $\mathcal{A}$ is a game, $\mathcal{A}^\perp$ has

$$\mathcal{W}_{\mathcal{A}^\perp}(x) = \mathcal{W}_{\mathcal{A}}(x)^\perp$$

If $\mathcal{A}$ and $\mathcal{B}$ are games with winnings, we define two games with arena $A \parallel B$:

$$\begin{array}{rcl}\mathcal{W}_{\mathcal{A}\otimes\mathcal{B}}(x_A \parallel x_B) & = & \mathcal{W}_{\mathcal{A}}(x_A) \wedge \mathcal{W}_{\mathcal{B}}(x_B) \\ \mathcal{W}_{\mathcal{A}\parr\mathcal{B}}(x_A \parallel x_B) & = & \mathcal{W}_{\mathcal{A}}(x_A) \vee \mathcal{W}_{\mathcal{B}}(x_B)\end{array}$$

with units $1 = (\varnothing, \mathcal{W}_1(\varnothing) = \top) \qquad \perp = (\varnothing, \mathcal{W}_\perp(\varnothing) = \perp)$

Winning strategies from $\mathcal{A}$ to $\mathcal{B}$ are winning Σ-strategies

$$\sigma : \mathcal{A}^\perp \parr \mathcal{B}$$

Lemma: $\otimes$, $\parr$, $^\perp$, $\odot$ preserve winning.

Herbrand's theorem, and Herbrand proofs

Herbrand's theorem (Buss?)

A formula φ is valid if and only if it has a **Herbrand proof***, i.e. if it has a valid* **substitution** *of a* **prenexification** *of a* **$\vee$-expansion**.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ \neg D(c) \vee D(y),\ \neg D(y) \vee D(z))}\ \textsc{Prop. Tautology}}{\vdash\ \neg D(c) \vee D(y), \exists x \forall y,\ \neg D(x) \vee D(y)}\ \exists_l, x := c, \forall_l}{\vdash \exists x \forall y\ \neg D(x) \vee D(y), \exists x \forall y\ \neg D(x) \vee D(y)}\ \exists_l, x := y, \forall_l}{\vdash \exists x \forall y\ \neg D(x) \vee D(y)}\ \textsc{Contraction}$$

1. **$\vee$-expansion.**
   $$(\exists x_1 \forall y_1\ \neg D(x_1) \vee D(y_2)) \quad \vee \quad (\exists x_2 \forall y_2\ \neg D(x_2) \vee D(y_2))$$
2. **Prenexification.**
   $$\exists x_1 \forall y_1 \exists x_2 \forall y_2\ (\neg D(x_1) \vee D(y_1)) \quad \vee \quad (\neg D(x_2) \vee D(y_2))$$
3. **Substitution** $\{x_1 := c, x_2 := y_1\}$
   $$\models (\neg D(c) \vee D(y_1)) \quad \vee \quad (\neg D(y_1) \vee D(y_2))$$

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But can we have a more intrinsic/geometric representation of Herbrand proofs?