The True Concurrency of Herbrand's Theorem

Aurore Alcolei
LIP & ENS Lyon

Joint work with

Pierre Clairambault, Martin Hyland, Glynn Winskel

CSL18, Birmingham

Roadmap

Herbrand's witnesses

Herbrand's theorem (Simple)

A purely existential formula $\exists \bar{x}\ \varphi(\bar{x})$ is valid in classical logic iff there is a finite set of witnesses $\bar{t}_1, \ldots, \bar{t}_n \in \mathsf{Tm}_\Sigma$ s.t. $\models \varphi(\bar{t}_1) \vee \ldots \vee \varphi(\bar{t}_n)$.

Example $\models \exists x\ \neg D(x) \vee D(f(x))$

$$\models (\neg D(c) \vee D(f(c))) \vee (\neg D(f(c)) \vee D(f(f(c))))$$

Herbrand's witnesses

Herbrand's theorem (Simple)

A purely existential formula $\exists \bar{x}\ \varphi(\bar{x})$ is valid in classical logic iff there is a finite set of witnesses $\bar{t}_1, \ldots, \bar{t}_n \in \mathsf{Tm}_\Sigma$ s.t. $\models \varphi(\bar{t}_1) \vee \ldots \vee \varphi(\bar{t}_n)$.

Example $\models \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x}))$

$$\models (\neg D(c) \vee D(f(c))) \vee (\neg D(f(c)) \vee D(f(f(c))))$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ \neg D(c) \vee D(f(c)),\ \neg D(f(c)) \vee D(f(f(c)))}\ \textsc{Prop. Tautology}}{\vdash\ \neg D(c) \vee D(f(c)),\ \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x}))}\ \exists_I, \mathrm{x} := f(c)}{\vdash \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x})),\ \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x}))}\ \exists_I, \mathrm{x} := c}{\vdash \exists \mathrm{x}\ \neg D(\mathrm{x}) \vee D(f(\mathrm{x}))}\ \textsc{Contraction}$$

Herbrand proofs

Herbrand's theorem (General)

A 1^{st} order formula φ is valid in classical logic iff it has a Herbrand proof.

Example $\models \exists x \forall y,\ \neg D(x) \vee D(y)$ (DF)

A proof for DF:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ \neg D(c) \vee D(y),\ \neg D(y) \vee D(z))}\ \textsc{Prop. Tautology}}{\vdash\ \neg D(c) \vee D(y), \exists x \forall y,\ \neg D(x) \vee D(y)}\ \exists_I, x := y, \forall_I}{\vdash \exists x \forall y\ \neg D(x) \vee D(y), \exists x \forall y\ \neg D(x) \vee D(y)}\ \exists_I, x := c, \forall_I}{\vdash \exists x \forall y\ \neg D(x) \vee D(y)}\ \textsc{Contraction}$$

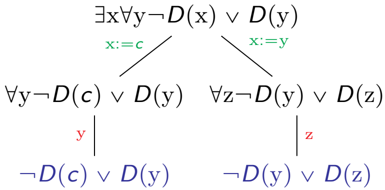
Herbrand proofs: Miller's **expansion trees**

Herbrand's theorem (Miller, 1987)

A 1^{st} order formula φ is valid in classical logic iff it has an expansion tree.

Example $\models \exists x \forall y, \neg D(x) \vee D(y)$ (DF)

An expansion tree for DF:



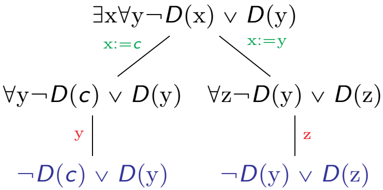
Herbrand proofs: Miller's **expansion trees**

Herbrand's theorem (Miller, 1987)

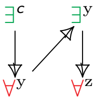
A 1^{st} *order formula* φ *is valid in classical logic iff it has an expansion tree.*

Example $\models \exists x \forall y,\ \neg D(x) \lor D(y)$ (DF)

An expansion tree for DF:



acyclicity



validity

$$\models (\neg D(c) \lor D(y)) \quad \lor \quad (\neg D(y) \lor D(z))$$

Herbrand proofs: Miller's **expansion trees**

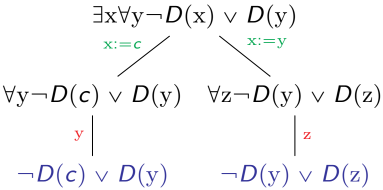
Herbrand's theorem (Miller, 1987)

A 1^{st} order formula φ is valid in classical logic iff it has an expansion tree.

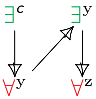
Proof: By translation from the cut-free sequent calculus. → not compositional.

Example $\models \exists x \forall y,\ \neg D(x) \vee D(y)$ (DF)

An expansion tree for DF:



acyclicity



validity

$$\models (\neg D(c) \vee D(y)) \vee (\neg D(y) \vee D(z))$$

Toward compositionality?

Question: find a **composable** notion of expansion tree/Herbrand proof?

Syntactic approaches: Heijltjes, McKinley, Hetzl and Weller, via notions of Herbrand proofs with cuts.

Contribution (semantic approach): Expansion trees as strategies in a concurrent game model (categories of winning Σ-strategies).

Herbrand's theorem (Compositional Herbrand's theorem)

A 1^{st} order formula φ is valid iff there is a winning Σ-strategy: $\sigma : [\![\varphi]\!]$.

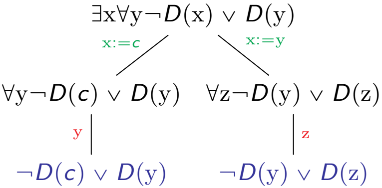
$$\frac{\vdash \Gamma, \varphi \qquad \vdash \Delta, \varphi^{\perp}}{\vdash \Gamma, \Delta}\ \textsc{Cut} \qquad\qquad \sigma = \sigma_1 \odot \sigma_2$$

Other related works: Games for first-order proofs (Laurent, Mimram)

Roadmap

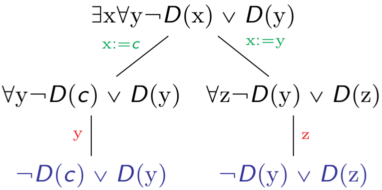
From expansion trees to concurrent strategies

An implicit two-player game played on the formula between ∃loïse and ∀bélard:

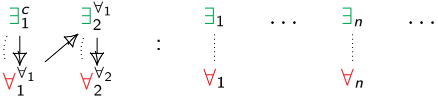


From expansion trees to concurrent strategies

An implicit two-player game played on the formula between ∃loïse and ∀bélard:

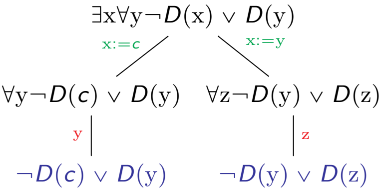


An interpretation of formulas as games and proofs as winning Σ-strategies:

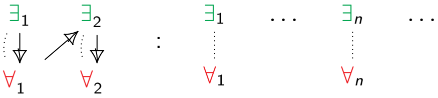


From expansion trees to concurrent strategies

An implicit two-player game played on the formula between ∃loïse and ∀bélard:



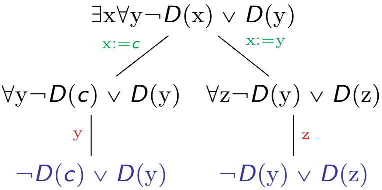
An interpretation of formulas as games and proofs as winning Σ-strategies:



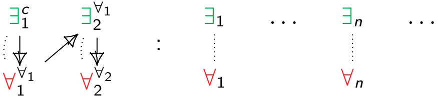
→ A causal game model

From expansion trees to concurrent strategies

An implicit two-player game played on the formula between ∃loïse and ∀bélard:



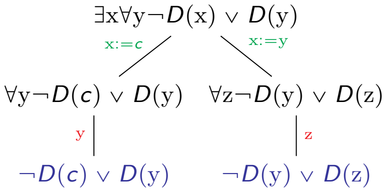
An interpretation of formulas as games and proofs as winning Σ-strategies:



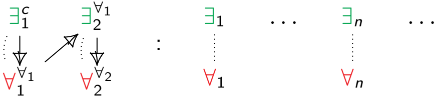
→ A causal game model with term labelling

From expansion trees to concurrent strategies

An implicit two-player game played on the formula between ∃loïse and ∀bélard:



An interpretation of formulas as games and proofs as winning Σ-strategies:



→ A causal game model with term labelling and winning conditions.

Concurrent arenas and strategies [RW]

Definition

A **arena** is a triple $(|A|, \leqslant_A, \mathrm{pol}_A)$, with:

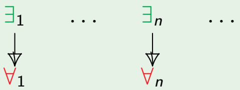
- $(|A|, \leqslant_A)$ a causal relation, i.e. a **partial order** with *finite* histories
- $\mathrm{pol}_A : |A| \to \{\forall, \exists\}$

Notation: $\mathcal{C}(A)$ is the set of configurations (down-closed subsets of A).

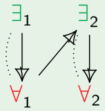
Definition

Strategies $\sigma : A$ are *certain* $(|\sigma|, \leqslant_\sigma)$, s.t. $\sigma \subseteq A$ and $\mathcal{C}(\sigma) \subseteq \mathcal{C}(A)$

The arena for $\exists x\ \forall y\ \psi(x, y)$



A strategy on $\exists x\ \forall y\ \psi(x, y)$



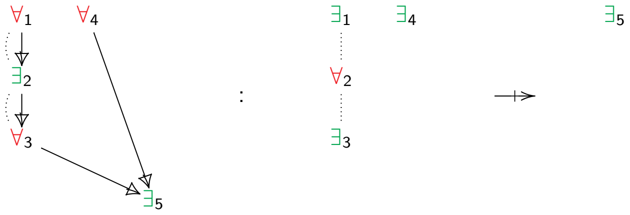
A (compact closed) category of arenas

Constructions on arenas.

- If A is an arena, $A^\perp$ has the same structure with polarity inverted.
- If A, B are arenas, $A \parallel B$ has events $|A| + |B|$, and components inherited.

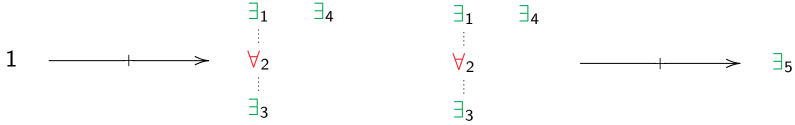
Definition

A **strategy from A to B** is $\sigma : A^\perp \parallel B$, written $\sigma : A \multimap B$.

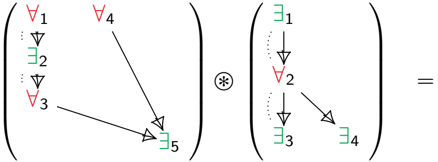


Composition $\tau \odot \sigma : A \multimap C$ is defined for all $\sigma : A \multimap B, \tau : B \multimap C$.

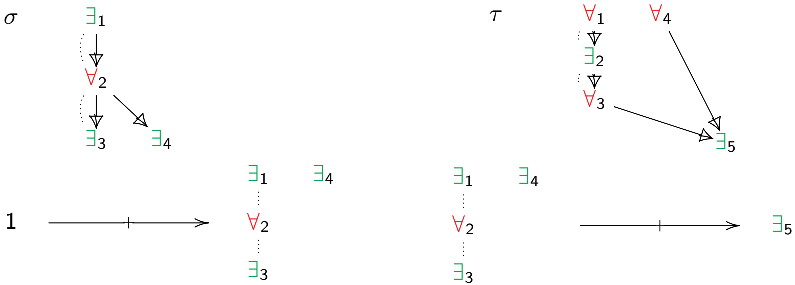
What is the result of the composition of the strategies σ and τ?



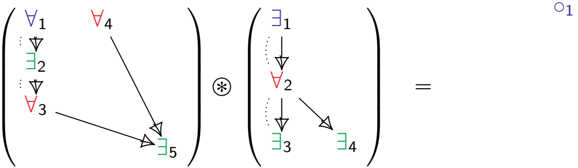
Interaction (a meet):



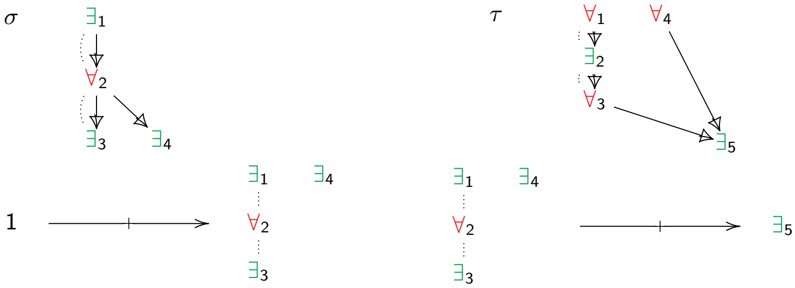
What is the result of the composition of the strategies σ and τ?



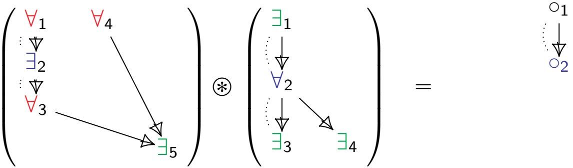
Interaction (a meet):



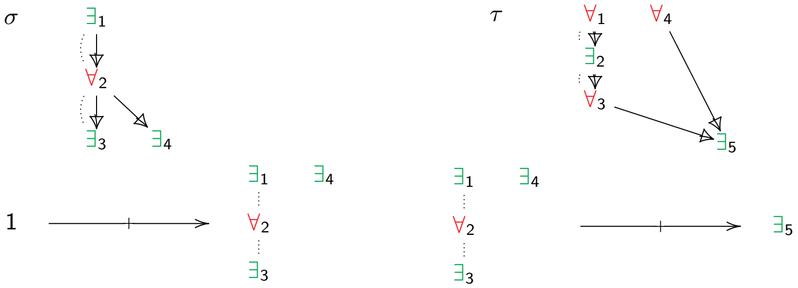
What is the result of the composition of the strategies σ and τ?



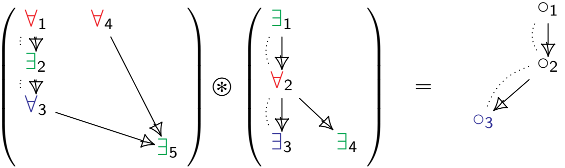
Interaction (a meet):



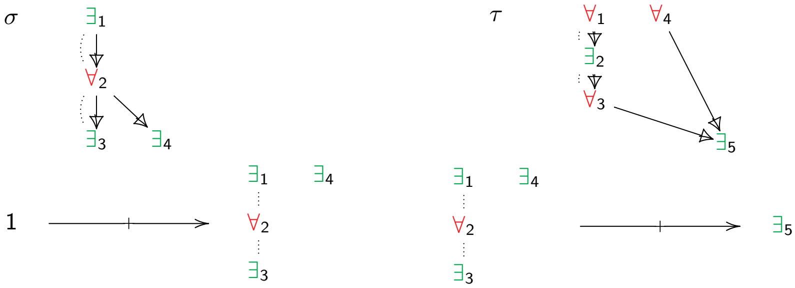
What is the result of the composition of the strategies σ and τ?



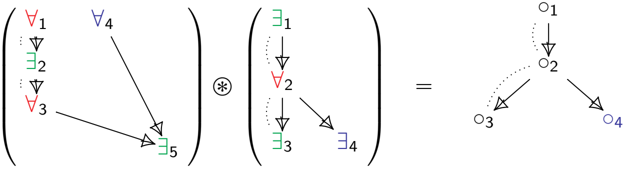
Interaction (a meet):



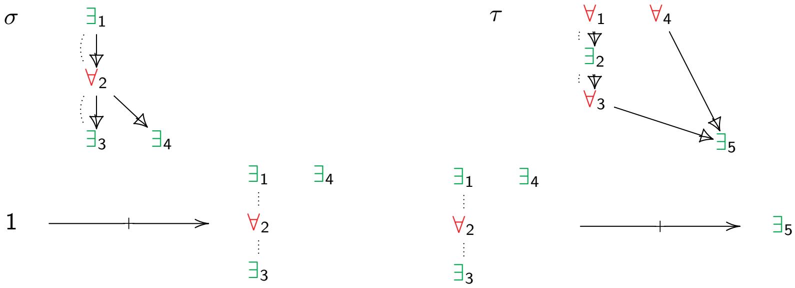
What is the result of the composition of the strategies σ and τ?



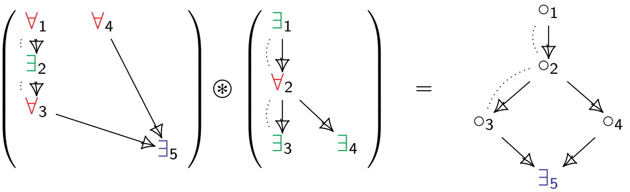
Interaction (a meet):



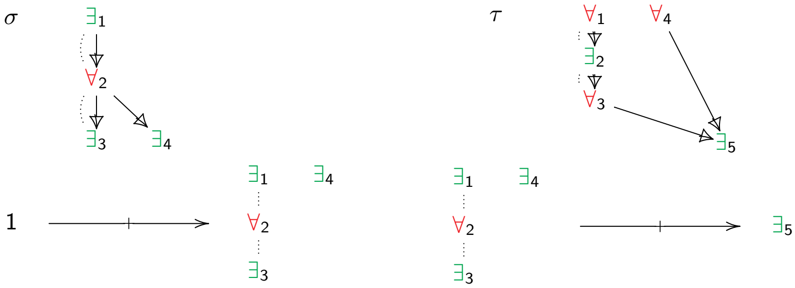
What is the result of the composition of the strategies σ and τ?



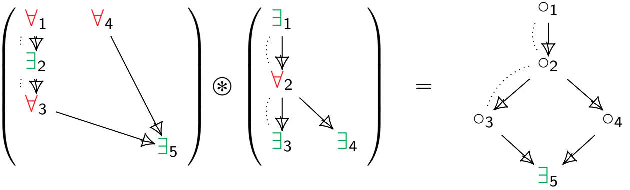
Interaction (a meet):



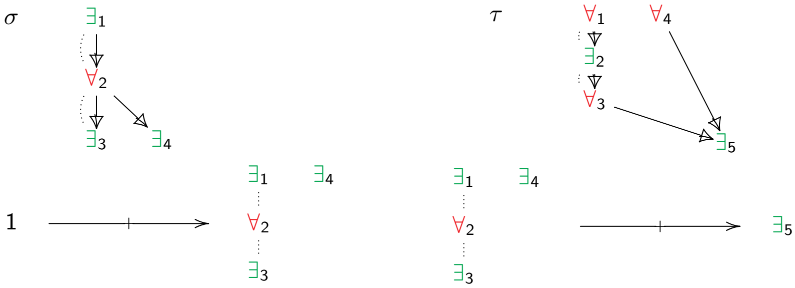
What is the result of the composition of the strategies σ and τ?



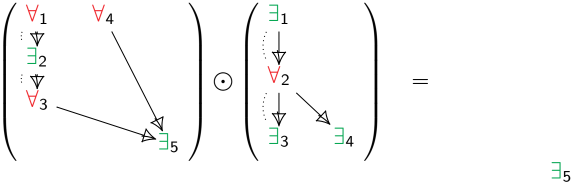
Interaction (a meet):



What is the result of the composition of the strategies σ and τ?



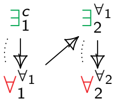
Composition (projection):



$\rightarrow$ A compact closed category CG.

Σ-strategies on arenas

A **strategy**, plus **free variables** (∀bélard's moves) and **terms** (∃loïse's moves).



Definition

A **Σ-strategy** on A is a strategy $\sigma : A$, with a **labeling function**

$$\lambda_\sigma : |\sigma| \to \mathsf{Tm}_\Sigma(|\sigma|)$$

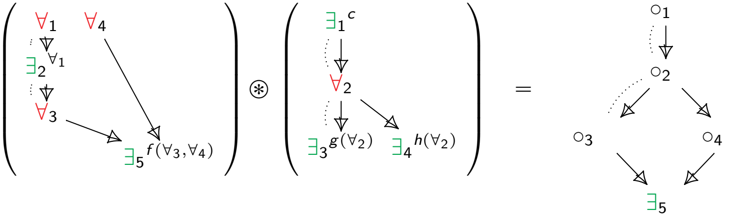
such that:

$$\begin{array}{lcl} \forall a^{\forall} \in |\sigma|, \quad \lambda_\sigma(a) & = & a \\ \forall a^{\exists} \in |\sigma|, \quad \lambda_\sigma(a) & \in & \mathsf{Tm}_\Sigma([a]^{\forall}_\sigma) \end{array}$$

where $[a]^{\forall}_\sigma = \{a' \in |\sigma| \mid a' \leqslant_\sigma a \ \&\ \mathrm{pol}_A(a') = \forall\}$.

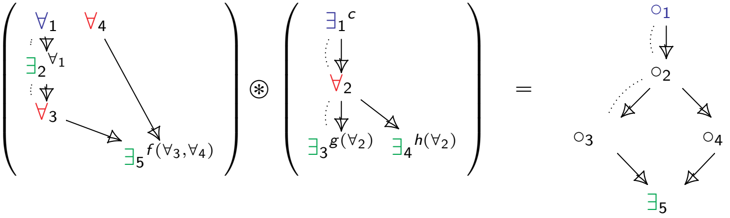
What is the result of the composition of the Σ-strategies σ and τ?

Same causal structure, with terms.



What is the result of the composition of the Σ-strategies σ and τ?

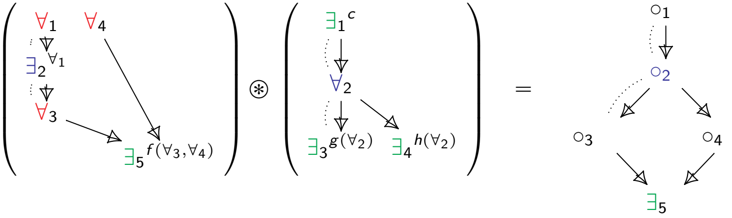
Same causal structure, with terms.



$$\circ_1 \doteq c$$

What is the result of the composition of the Σ-strategies σ and τ?

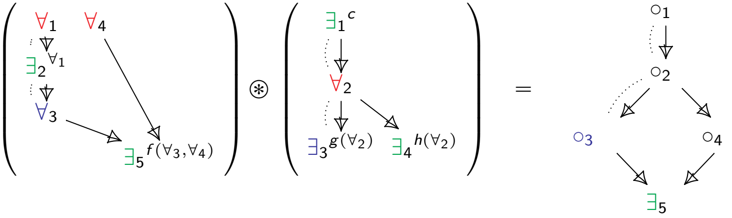
Same causal structure, with terms.



$$\circ_1 \doteq c$$
$$\circ_1 \doteq \circ_2$$

What is the result of the composition of the Σ-strategies σ and τ?

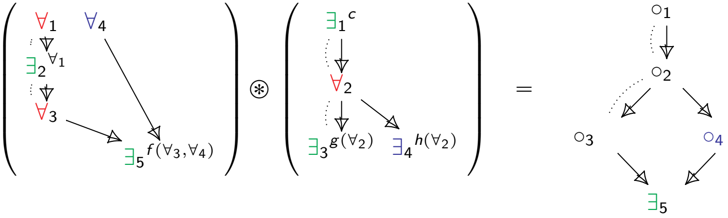
Same causal structure, with terms.



$$\circ_1 \doteq c$$
$$\circ_1 \doteq \circ_2$$
$$\circ_3 \doteq g(\circ_2)$$

What is the result of the composition of the Σ-strategies σ and τ?

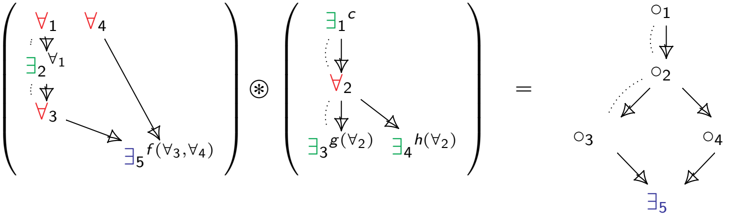
Same causal structure, with terms.



$$
\begin{aligned}
\circ_1 &\doteq c \\
\circ_1 &\doteq \circ_2 \\
\circ_3 &\doteq g(\circ_2) \\
\circ_4 &\doteq h(\circ_2)
\end{aligned}
$$

What is the result of the composition of the Σ-strategies σ and τ?

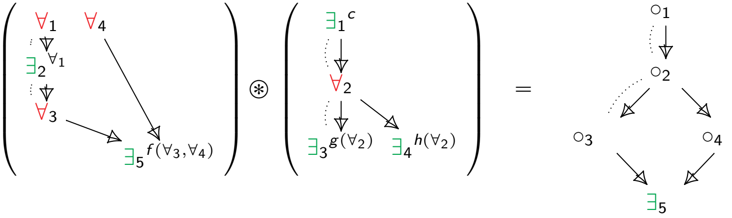
Same causal structure, with terms.



$$
\begin{aligned}
\circ_1 &\doteq c \\
\circ_1 &\doteq \circ_2 \\
\circ_3 &\doteq g(\circ_2) \\
\circ_4 &\doteq h(\circ_2) \\
f(\circ_3, \circ_4) &\doteq \exists_5
\end{aligned}
$$

What is the result of the composition of the Σ-strategies σ and τ?

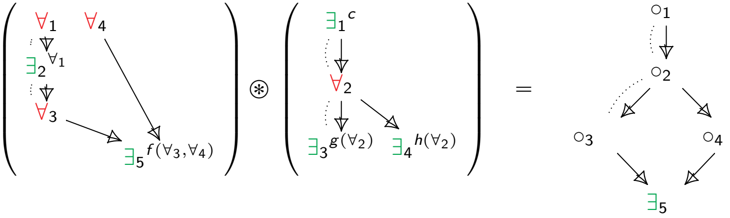
Same causal structure, with terms.



$$\left\{ \begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array} \right\}$$

What is the result of the composition of the Σ-strategies σ and τ?

Same causal structure, with terms.

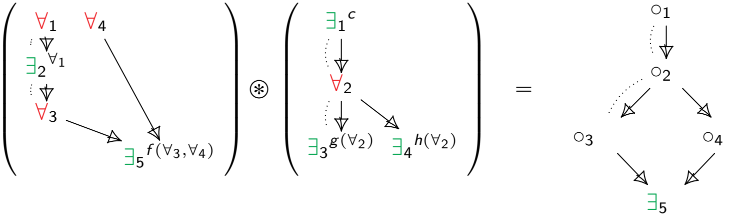


$$\left\{ \begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array} \right\}$$

with **m.g.u.** $\left\{ \quad \right\}$

What is the result of the composition of the Σ-strategies σ and τ?

Same causal structure, with terms.



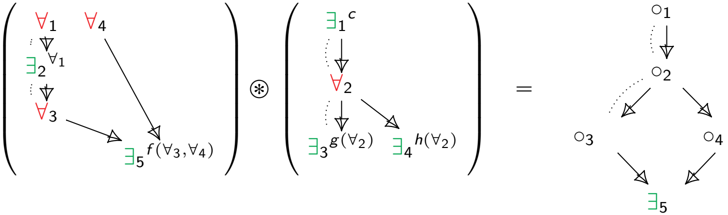
$$\left\{ \begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array} \right\}$$

with **m.g.u.**

$$\left\{ \begin{array}{rcl} \circ_1 & \mapsto & c \end{array} \right\}$$

What is the result of the composition of the Σ-strategies σ and τ?

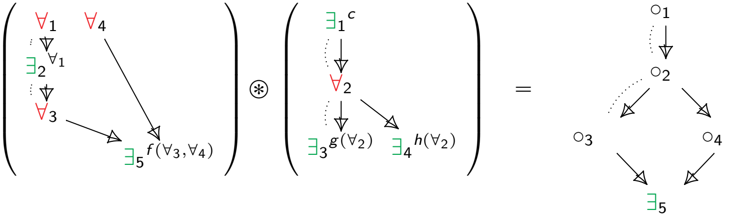
Same causal structure, with terms.



$$\left\{ \begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array} \right\} \quad \text{with } \textbf{m.g.u.} \quad \left\{ \begin{array}{rcl} \circ_1 & \mapsto & c \\ \circ_2 & \mapsto & c \end{array} \right\}$$

What is the result of the composition of the Σ-strategies σ and τ?

Same causal structure, with terms.



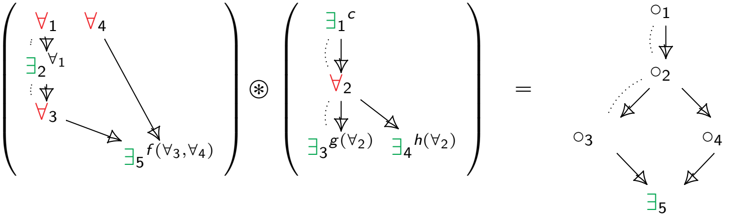
$$\left\{ \begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array} \right\}$$

with **m.g.u.**

$$\left\{ \begin{array}{lcl} \circ_1 & \mapsto & c \\ \circ_2 & \mapsto & c \\ \circ_3 & \mapsto & g(c) \\ & & \\ & & \end{array} \right\}$$

What is the result of the composition of the Σ-strategies σ and τ?

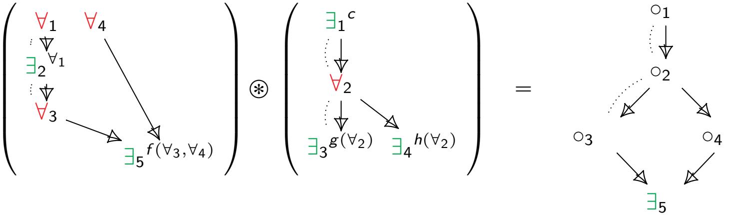
Same causal structure, with terms.



$$\left\{\begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array}\right\} \quad \text{with } \textbf{m.g.u.} \quad \left\{\begin{array}{lcl} \circ_1 & \mapsto & c \\ \circ_2 & \mapsto & c \\ \circ_3 & \mapsto & g(c) \\ \circ_4 & \mapsto & h(c) \end{array}\right\}$$

What is the result of the composition of the Σ-strategies σ and τ?

Same causal structure, with terms.



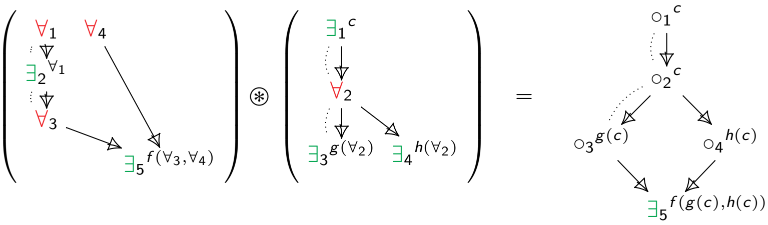
$$\left\{\begin{array}{ccc} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array}\right\}$$

with **m.g.u.**

$$\left\{\begin{array}{ccc} \circ_1 & \mapsto & c \\ \circ_2 & \mapsto & c \\ \circ_3 & \mapsto & g(c) \\ \circ_4 & \mapsto & h(c) \\ \exists_5 & \mapsto & f(g(c), h(c)) \end{array}\right\}$$

What is the result of the composition of the Σ-strategies σ and τ?

Same causal structure, with terms.



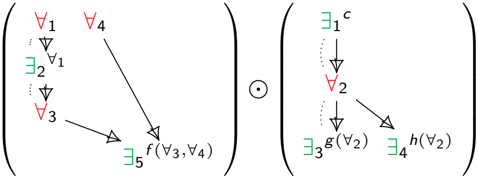
$$\left\{ \begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array} \right\}$$

with **m.g.u.**

$$\left\{ \begin{array}{lcl} \circ_1 & \mapsto & c \\ \circ_2 & \mapsto & c \\ \circ_3 & \mapsto & g(c) \\ \circ_4 & \mapsto & h(c) \\ \exists_5 & \mapsto & f(g(c), h(c)) \end{array} \right\}$$

What is the result of the composition of the Σ-strategies σ and τ?

Same causal structure, with terms.



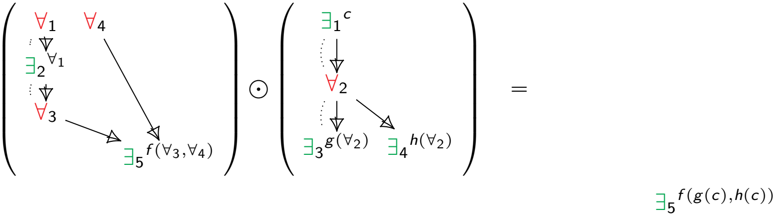
$=$

$\exists_5{}^{f(g(c),h(c))}$

$$\left\{ \begin{array}{ccc} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array} \right\} \quad \text{with } \textbf{m.g.u.} \quad \left\{ \begin{array}{ccc} \circ_1 & \mapsto & c \\ \circ_2 & \mapsto & c \\ \circ_3 & \mapsto & g(c) \\ \circ_4 & \mapsto & h(c) \\ \exists_5 & \mapsto & f(g(c), h(c)) \end{array} \right\}$$

What is the result of the composition of the Σ-strategies σ and τ?

Same causal structure, with terms.

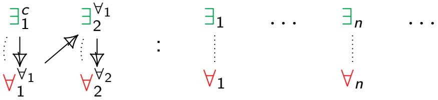


$$\left\{ \begin{array}{rcl} \circ_1 & \doteq & c \\ \circ_1 & \doteq & \circ_2 \\ \circ_3 & \doteq & g(\circ_2) \\ \circ_4 & \doteq & h(\circ_2) \\ f(\circ_3, \circ_4) & \doteq & \exists_5 \end{array} \right\} \quad \text{with } \textbf{m.g.u.} \quad \left\{ \begin{array}{lcl} \circ_1 & \mapsto & c \\ \circ_2 & \mapsto & c \\ \circ_3 & \mapsto & g(c) \\ \circ_4 & \mapsto & h(c) \\ \exists_5 & \mapsto & f(g(c), h(c)) \end{array} \right\}$$

→ A new compact closed category Σ-CG.

Example of winning conditions

Consider the Σ-strategy $\sigma : [\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$ over DF

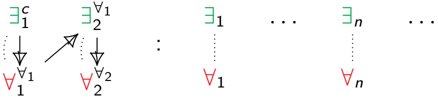


Validity in expansion trees:

$$\models (\neg D(c) \vee D(\forall_1)) \quad \vee \quad (\neg D(\forall_1) \vee D(\forall_2))$$

Example of winning conditions

Consider the Σ-strategy $\sigma : [\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$ over DF



Validity in expansion trees:

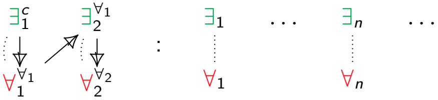
$$\models (\neg D(c) \vee D(\forall_1)) \quad \vee \quad (\neg D(\forall_1) \vee D(\forall_2))$$

Can be decomposed into

$$\models \quad (\neg D(\exists_1) \vee D(\forall_1)) \quad \vee \quad (\neg D(\exists_2) \vee D(\forall_2)) \quad [\exists_1 \mapsto c; \exists_2 \mapsto \forall_1]$$

Example of winning conditions

Consider the Σ-strategy $\sigma : [\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$ over DF



Validity in expansion trees:

$$\models (\neg D(c) \vee D(\forall_1)) \quad \vee \quad (\neg D(\forall_1) \vee D(\forall_2))$$

Can be decomposed into

$$\models \quad \underbrace{(\neg D(\exists_1) \vee D(\forall_1)) \quad \vee \quad (\neg D(\exists_2) \vee D(\forall_2))}_{\textbf{Winning conditions},\ \mathcal{W}_{\mathrm{DF}}(|\sigma|)} \quad \underbrace{[\exists_1 \mapsto c; \exists_2 \mapsto \forall_1]}_{\text{Labelling},\ \lambda_\sigma}$$

Winning conditions on arenas

Definition

A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition. σ is a **winning on** x if $\models \mathcal{W}_A(x)[\lambda_\sigma]$.

To each configuration of $[\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$, we associate a **formula**:

$$\begin{matrix} \exists_1 & \exists_2 & \dots \\ \vdots & \vdots & \\ \forall_1 & \forall_2 & \end{matrix}$$

Winning conditions on arenas

Definition

A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition. σ is a **winning on x** if $\models \mathcal{W}_{A}(x)[\lambda_{\sigma}]$.

To each configuration of $[\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$, we associate a **formula**:

$\mapsto$ $\bot$

Winning conditions on arenas

Definition

A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition. σ is a **winning on** x if $\models \mathcal{W}_A(x)[\lambda_\sigma]$.

To each configuration of $[\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$, we associate a **formula**:

$\exists_1$ $\mapsto$ $\top$

Winning conditions on arenas

Definition

A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition. σ is a **winning on** x if $\models \mathcal{W}_A(x)[\lambda_\sigma]$.

To each configuration of $[\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$, we associate a **formula**:

$\exists_1$

$\vdots$

$\forall_1$ $\quad\mapsto\quad$ $\neg D(\exists_1) \vee D(\forall_1)$

Winning conditions on arenas

Definition

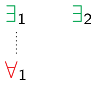
A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition. σ is a **winning on x** if $\models \mathcal{W}_A(x)[\lambda_\sigma]$.

To each configuration of $[\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$, we associate a **formula**:



$$\mapsto \qquad (\neg D(\exists_1) \vee D(\forall_1)) \quad \vee \quad \top$$

Winning conditions on arenas

Definition

A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition. σ is a **winning on x** if $\models \mathcal{W}_A(x)[\lambda_\sigma]$.

To each configuration of $[\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$, we associate a **formula**:

$$\begin{matrix} \exists_1 & \exists_2 \\ \vdots & \vdots \\ \forall_1 & \forall_2 \end{matrix} \quad \mapsto \quad (\neg D(\exists_1) \vee D(\forall_1)) \ \vee \ (\neg D(\exists_2) \vee D(\forall_2))$$

Winning conditions on arenas

Definition

A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition. σ is a **winning on** x if $\models \mathcal{W}_A(x)[\lambda_\sigma]$.

To each configuration of $[\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$, we associate a **formula**:

$$\begin{matrix} \exists_1 & \exists_2 & \dots \\ \vdots & \vdots & \\ \forall_1 & \forall_2 & \end{matrix} \quad \mapsto \quad (\neg D(\exists_1) \vee D(\forall_1)) \quad \vee \quad (\neg D(\exists_2) \vee D(\forall_2)) \quad \vee \quad \dots$$

Winning conditions on arenas

Definition

A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition. σ is a **winning on** x if $\models \mathcal{W}_A(x)[\lambda_\sigma]$.

To each configuration of $[\![\exists \mathrm{x}\ \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})]\!]$, we associate a **formula**:

$$\begin{array}{ccc} \exists_1 & \exists_2 & \dots \\ \vdots & \vdots & \\ \forall_1 & \forall_2 & \end{array} \quad \mapsto \quad (\neg D(\exists_1) \vee D(\forall_1)) \quad \vee \quad (\neg D(\exists_2) \vee D(\forall_2)) \quad \vee \quad \dots$$

Winning conditions on arenas

Definition

A **game** $\mathcal{A}$ is an arena A, together with winning conditions:

$$\mathcal{W}_{\mathcal{A}} : (x \in \mathcal{C}(A)) \mapsto \mathrm{QF}_{\Sigma}(x)$$

where $\mathrm{QF}_{\Sigma}(x)$ is the set of **quantifier-free** formulas on signature Σ and free variables in x, extended with **countable** conjunctions and disjunctions.

Definition

A Σ-strategy $\sigma : A$ is **winning on $\mathcal{W}_{\mathcal{A}}$** iff for all $x \in \mathcal{C}^{\infty}(\sigma)$ $\exists$-**maximal**,

$$\models \mathcal{W}_{A}(x)[\lambda_{\sigma}]$$

- → Two new constructors on games: $\otimes$ (conjunction) and $⅋$ (disjunction)
- → Winning strategies $\sigma : \mathcal{A}^{\perp} ⅋ \mathcal{B}$ are **stable under composition ($\star$-autonomous category).**

Roadmap

The interpretation in a nutshell

Propositional connectives (MLL $\star$-autonomous model)

Quantifiers

$$[\![\exists \mathrm{x}\varphi]\!]_{\mathcal{V}} = \quad \exists_{\mathrm{x}}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}} \qquad\qquad [\![\forall \mathrm{x}\varphi]\!]_{\mathcal{V}} = \quad \forall_{\mathrm{x}}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}}$$

$\longrightarrow$ A model for proofs of first order MLL

The interpretation in a nutshell

Propositional connectives (MLL $\star$-autonomous model)

Quantifiers and exponentials

$$[\![\exists \mathrm{x}\varphi]\!]_{\mathcal{V}} = ?\ \exists_{\mathrm{x}}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}} \qquad [\![\forall \mathrm{x}\varphi]\!]_{\mathcal{V}} = \forall_{\mathrm{x}}.[\![\varphi]\!]_{\mathcal{V}\uplus\{x\}}$$

$$?\ A = \|_{n\in\omega} A$$

$$\mathcal{W}_{?\mathcal{A}}(\|_i\ x_i) = \bigvee_i \mathcal{W}_{\mathcal{A}}(x_i)$$

$\rightarrow$ A model for proofs of first order MLL

The interpretation in a nutshell

Propositional connectives (MLL $\star$-autonomous model)

Quantifiers and exponentials

$$[\![\exists x \varphi]\!]_{\mathcal{V}} = ?\ \exists_{x}.[\![\varphi]\!]_{\mathcal{V} \uplus \{x\}} \qquad [\![\forall x \varphi]\!]_{\mathcal{V}} = !\ \forall_{x}.[\![\varphi]\!]_{\mathcal{V} \uplus \{x\}}$$

$$?\ A = \|_{n \in \omega} A \qquad !\ A = \|_{n \in \omega} A$$

$$\mathcal{W}_{?\mathcal{A}}(\|_i\ x_i) = \bigvee_i \mathcal{W}_{\mathcal{A}}(x_i) \qquad \mathcal{W}_{!\mathcal{A}}(\|_i\ x_i) = \bigwedge_i \mathcal{W}_{\mathcal{A}}(x_i)$$

$\longrightarrow$ A model for proofs of first order MLL

The interpretation in a nutshell

Propositional connectives (MLL $\star$-autonomous model)

Quantifiers and exponentials

$$[\![\exists x \varphi]\!]_{\mathcal{V}} = ?\ \exists_x . [\![\varphi]\!]_{\mathcal{V} \uplus \{x\}} \qquad [\![\forall x \varphi]\!]_{\mathcal{V}} = !\ \forall_x . [\![\varphi]\!]_{\mathcal{V} \uplus \{x\}}$$

$$?\ A = \|_{n \in \omega} A \qquad !\ A = \|_{n \in \omega} A$$

$$\mathcal{W}_{?\mathcal{A}}(\|_i\ x_i) = \bigvee_i \mathcal{W}_{\mathcal{A}}(x_i) \qquad \mathcal{W}_{!\mathcal{A}}(\|_i\ x_i) = \bigwedge_i \mathcal{W}_{\mathcal{A}}(x_i)$$

Contraction : for any formula φ, $c_{[\![\varphi]\!]} : [\![\varphi]\!] \twoheadrightarrow [\![\varphi]\!] \otimes [\![\varphi]\!]$

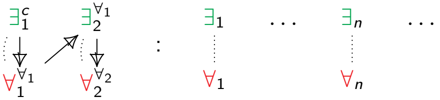
$\rightarrow$ A model for proofs of first order MLL + contraction (= first order LK)

Herbrand's theorem

Herbrand's theorem (Compositional Herbrand's theorem)

A 1^{st} *order formula* φ *is valid iff there is a winning* Σ*-strategy:* $\sigma : [\![\varphi]\!]$.

$[\![\pi]\!] : [\![\exists x\ \forall y\ \neg D(x) \vee D(y)]\!] =$



On cut free proofs $\sim$ Expansion trees with explicit acyclicity witness

Conclusion

Herbrand's theorem (Compositional Herbrand's theorem)

A 1^{st} order formula φ is valid iff there is a winning Σ-strategy: $\sigma : [\![\varphi]\!]$.

Games$_\Sigma$: a concurrent game model with **terms** and **winnings**

Proof:
a framework to interpret **first order proofs** (with extra constructors $\forall, \exists, !, ?$)

- $\longrightarrow$ Does not preserve cuts elimination in LK (by necessity)
- $\longrightarrow$ Reflects some dynamics of LK: infinite strategies.

Future investigation: a finatary composition of strategies?

Herbrand's theorem, and Herbrand proofs

Herbrand's theorem (Buss?)

A formula φ is valid if and only if it has a **Herbrand proof**, *i.e. if it has a valid* **substitution** *of a* **prenexification** *of a* $\vee$**-expansion**.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ \neg D(\mathrm{c}) \vee D(\mathrm{y}),\ \neg D(\mathrm{y}) \vee D(\mathrm{z}))}\ \textsc{Prop. Tautology}}{\vdash\ \neg D(\mathrm{c}) \vee D(\mathrm{y}), \exists \mathrm{x} \forall \mathrm{y},\ \neg D(\mathrm{x}) \vee D(\mathrm{y})}\ \exists_I, \mathrm{x} := c, \forall_I}{\vdash \exists \mathrm{x} \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y}), \exists \mathrm{x} \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})}\ \exists_I, \mathrm{x} := y, \forall_I}{\vdash \exists \mathrm{x} \forall \mathrm{y}\ \neg D(\mathrm{x}) \vee D(\mathrm{y})}\ \textsc{Contraction}$$

1. $\vee$**-expansion.**

   $$(\exists \mathrm{x}_1 \forall \mathrm{y}_1\ \neg D(\mathrm{x}_1) \vee D(\mathrm{y}_2)) \quad \vee \quad (\exists \mathrm{x}_2 \forall \mathrm{y}_2\ \neg D(\mathrm{x}_2) \vee D(\mathrm{y}_2))$$

2. **Prenexification.**

   $$\exists \mathrm{x}_1 \forall \mathrm{y}_1 \exists \mathrm{x}_2 \forall \mathrm{y}_2\ (\neg D(\mathrm{x}_1) \vee D(\mathrm{y}_1)) \quad \vee \quad (\neg D(\mathrm{x}_2) \vee D(\mathrm{y}_2))$$

3. **Substitution** $\{\mathrm{x}_1 := c, \mathrm{x}_2 := \mathrm{y}_1\}$

   $$\models (\neg D(c) \vee D(\mathrm{y}_1)) \quad \vee \quad (\neg D(\mathrm{y}_1) \vee D(\mathrm{y}_2))$$

Herbrand's theorem, and Herbrand proofs

Herbrand's theorem (Buss?)

A formula φ is valid if and only if it has a **Herbrand proof**, *i.e. if it has a valid* **substitution** *of a* **prenexification** *of a* $\vee$**-expansion**.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ \neg D(c)\vee D(y),\ \neg D(y)\vee D(z))}\text{ Prop. Tautology}}{\vdash\ \neg D(c)\vee D(y), \exists x\forall y,\ \neg D(x)\vee D(y)}\ \exists_I, x:=c, \forall_I}{\vdash \exists x\forall y\ \neg D(x)\vee D(y), \exists x\forall y\ \neg D(x)\vee D(y)}\ \exists_I, x:=y, \forall_I}{\vdash \exists x\forall y\ \neg D(x)\vee D(y)}\ \text{Contraction}$$

1. $\vee$**-expansion.**
   $$(\exists x_1\forall y_1\ \neg D(x_1)\vee D(y_2)) \quad \vee \quad (\exists x_2\forall y_2\ \neg D(x_2)\vee D(y_2))$$
2. **Prenexification.**
   $$\exists x_1\forall y_1\exists x_2\forall y_2\ (\neg D(x_1)\vee D(y_1)) \quad \vee \quad (\neg D(x_2)\vee D(y_2))$$
3. **Substitution** $\{x_1 := c, x_2 := y_1\}$
   $$\models (\neg D(c)\vee D(y_1)) \quad \vee \quad (\neg D(y_1)\vee D(y_2))$$

But can we have a more intrinsic/geometric representation of Herbrand proofs?

Composition of plain strategies

Interaction. An **elementary event structure** is a partial order $(|\mathbf{q}|, \leqslant_{\mathbf{q}})$ such that for any $e \in |\mathbf{q}|$, $[e]_{\mathbf{q}}$ is finite.

Proposition

For $\mathbf{q}, \mathbf{q}'$, we say that

$$\mathbf{q} \preccurlyeq \mathbf{q}' \quad \Leftrightarrow \quad |\mathbf{q}| \subseteq |\mathbf{q}'| \;\&\; \mathcal{C}^\infty(\mathbf{q}) \subseteq \mathcal{C}^\infty(\mathbf{q}')$$

Then any two $\mathbf{q}$, $\mathbf{q}'$ have a greatest lower bound (meet-semilattice).

For $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$, define (ignoring polarities)

$$\tau \circledast \sigma = (\sigma \parallel C) \wedge (A \parallel \tau)$$

Composition. Define

$$\tau \odot \sigma \quad = \quad \tau \circledast \sigma \downarrow A, C \quad : \quad A \mathrel{\mathrlap{\rightarrow}\mkern1mu{+}}\!\!\!> C$$

This is composition in Rideau and Winskel's concurrent games, simplified.

Interpreting ∀I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}} \parr \qquad [\![\varphi]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}}$$

Interpreting $\forall$I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}} \parr \color{red}{\forall \mathrm{x}.}\ [\![\varphi]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}}$$

Interpreting $\forall$I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}} \qquad \parr\ \forall \mathrm{x}.\ [\![\varphi]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}}$$

Interpreting ∀I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}} \qquad \invamp\ \forall \mathrm{x}.\ [\![\varphi]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}}$$

All $\exists^{t}$ moves where $\mathrm{x} \in \mathrm{fv}(t)$ are set to depend on $\forall$.

Interpreting ∀I

In a linear setting for now.

$$\forall\mathrm{I}\ \frac{\vdash^{\mathcal{V}\uplus\{\mathrm{x}\}} \Gamma, \varphi}{\vdash^{\mathcal{V}} \Gamma, \forall \mathrm{x}.\, \varphi}\ \mathrm{x} \notin \mathcal{V}, \mathrm{x} \notin \mathrm{fv}(\Gamma)$$

$$\sigma : [\![\Gamma]\!]_{\mathcal{V}} \qquad \parr\ \forall \mathrm{x}.\ [\![\varphi]\!]_{\mathcal{V}\uplus\{\mathrm{x}\}}$$

All $\exists^{t}$ moves where $\mathrm{x} \in \mathrm{fv}(t)$ are set to depend on $\forall$.

The variable x is replaced by $\forall$ in λ_σ.

Interpreting ∃I

$$\exists\mathrm{I}\ \frac{\vdash^{\mathcal{V}} \Gamma, \varphi[t/\mathrm{x}]}{\vdash^{\mathcal{V}} \Gamma, \exists \mathrm{x}.\, \varphi}\ t \in \mathsf{Tm}_{\Sigma}(\mathcal{V})$$

Composition with the winning Σ-strategy

$$u_{\mathcal{A},t} : \mathcal{A}[t/\mathrm{x}] \overset{\mathcal{V}\text{-Games}}{\longrightarrow} \exists\mathrm{x}\mathcal{A}$$

playing $\exists^t$, then copycat on A.

Interpreting the proof of the drinker's formula

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y})}{\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})}}{\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}}{\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))}$$

$$(\neg D(c),\ D(\mathrm{y}),\ \neg D(\mathrm{y}),\ \forall \mathrm{y} D(\mathrm{y}))$$

$\forall_2$

Interpreting the proof of the drinker's formula

$$\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y})$$

$$\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})$$

$$\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

$$\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

$$\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

$$\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

$$\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

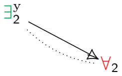
$$(\neg D(c) \,,\; D(\mathrm{y}) \,,\; (D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})))$$

$\forall_2$

Interpreting the proof of the drinker's formula

$$
\begin{array}{c}
\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y}) \\
\hline
\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}) \\
\hline
\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))
\end{array}
$$

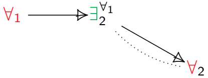
$$(\neg D(c), \quad D(\mathrm{y}) \quad , \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})))$$



Interpreting the proof of the drinker's formula

$$
\begin{array}{c}
\vdash^{y} \neg D(c), D(y), \neg D(y), \forall y D(y) \\
\hline
\vdash^{y} \neg D(c), D(y), D(y) \Rightarrow \forall y D(y) \\
\hline
\vdash^{y} \neg D(c), D(y), \exists x(D(x) \Rightarrow \forall y D(y)) \\
\hline
\vdash \neg D(c), \forall y D(y), \exists x(D(x) \Rightarrow \forall y D(y)) \\
\hline
\vdash D(c) \Rightarrow \forall y D(y), \exists x(D(x) \Rightarrow \forall y D(y)) \\
\hline
\vdash \exists x(D(x) \Rightarrow \forall y D(y)), \exists x(D(x) \Rightarrow \forall y D(y)) \\
\hline
\vdash \exists x(D(x) \Rightarrow \forall y D(y))
\end{array}
$$

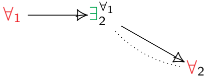
$\neg D(c)$, $\forall y D(y)$, $\exists x(D(x) \Rightarrow \forall y D(y))$



Interpreting the proof of the drinker's formula

$$
\begin{array}{c}
\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y}) \\
\hline
\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}) \\
\hline
\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))
\end{array}
$$

$$D(c) \Rightarrow \quad \forall \mathrm{y} D(\mathrm{y}) \;,\; \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$



Interpreting the proof of the drinker's formula

$$\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y})$$

$$\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})$$

$$\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

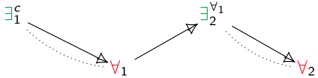
$$\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

$$\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

$$\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

$$\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$

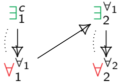
$$\exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \quad \forall \mathrm{y} D(\mathrm{y})) \;,\; \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$



Interpreting the proof of the drinker's formula

$$
\begin{array}{c}
\vdash^{y} \neg D(c), D(\mathrm{y}), \neg D(\mathrm{y}), \forall \mathrm{y} D(\mathrm{y}) \\
\hline
\vdash^{y} \neg D(c), D(\mathrm{y}), D(\mathrm{y}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}) \\
\hline
\vdash^{y} \neg D(c), D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \neg D(c), \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash D(c) \Rightarrow \forall \mathrm{y} D(\mathrm{y}), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})), \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y})) \\
\hline
\vdash \exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))
\end{array}
$$

$$\exists \mathrm{x}(D(\mathrm{x}) \Rightarrow \forall \mathrm{y} D(\mathrm{y}))$$



Composition of winning strategies

Constructions. If $\mathcal{A}$ is a game, $\mathcal{A}^\perp$ has

$$\mathcal{W}_{\mathcal{A}^\perp}(x) = \mathcal{W}_{\mathcal{A}}(x)^\perp$$

If $\mathcal{A}$ and $\mathcal{B}$ are games with winnings, we define two games with arena $A \parallel B$:

$$\begin{array}{rcl} \mathcal{W}_{\mathcal{A}\otimes\mathcal{B}}(x_A \parallel x_B) & = & \mathcal{W}_{\mathcal{A}}(x_A) \wedge \mathcal{W}_{\mathcal{B}}(x_B) \\ \mathcal{W}_{\mathcal{A}⅋\mathcal{B}}(x_A \parallel x_B) & = & \mathcal{W}_{\mathcal{A}}(x_A) \vee \mathcal{W}_{\mathcal{B}}(x_B) \end{array}$$

with units $1 = (\varnothing, \mathcal{W}_1(\varnothing) = \top) \qquad \perp = (\varnothing, \mathcal{W}_\perp(\varnothing) = \perp)$

Winning strategies from $\mathcal{A}$ to $\mathcal{B}$ are winning Σ-strategies

$$\sigma : \mathcal{A}^\perp ⅋ \mathcal{B}$$

Lemma: $\otimes$, ⅋, $^\perp$, $\odot$ preserve winning.

Proposition

*There is a $\star$-**autonomous** category* Games_Σ *with games as objects, and winning Σ-strategies as morphisms.*