CTL, the branching-time temporal logic

Cătălin Dima

Université Paris-Est Créteil

Temporal properties

- Safety, termination, mutual exclusion – LTL.
- Liveness, reactiveness, responsiveness, infinitely repeated behaviors – LTL.
- Available choices, strategies, adversial situations?

CNIL

Tout utilisateur peut demander le retrait de ses données...

- How do we interpret peut?
  - $p =$ demander le retrait...
  - Then formula = $\Box p$??
  - NO!

Strategy to win a game

Black has a strategy to put the game in a situation from which White king will never get close to Black pawn.

- Not specifiable in LTL either!

Computational Tree Logic (CTL)

Syntax:

$$\Phi ::= p \mid \Phi \wedge \Phi \mid \neg\Phi \mid \forall\bigcirc\Phi \mid \forall\Box\Phi \mid \forall(\Phi\,\mathcal{U}\,\Phi) \mid \exists\bigcirc\Phi \mid \exists\Box\Phi \mid \exists(\Phi\,\mathcal{U}\,\Phi)$$

- Grammar for the logic: the set of formulas is the set of "words" obtained by this (context-free!) grammar, with Φ viewed as nonterminal.
- Syntactic tree for each formula.
  - $\forall, \exists$: path quantifier (will see why!).
  - $\mathcal{U}, \Box, \Diamond$: temporal quantifiers.
  - Alternative notations (for the temporal operators): $\Box\phi = G\phi$, $\Diamond\phi = F\phi$, $\bigcirc\phi = X\phi$.
  - Each path quantifier must be followed by a temporal quantifier in the syntactic tree of each formula.
- Sample formula: $p \wedge \exists\Box(\neg\forall\bigcirc p \vee \forall(p\,\mathcal{U}(\neg q \wedge \exists\bigcirc q)))$.
  - Draw its syntactic tree!
- Strict alternation:
  - A non-CTL formula $p \wedge \exists\Box(\neg\forall\bigcirc p \vee (p\,\mathcal{U}(\neg q \wedge \exists\bigcirc q)))$.
  - ... because the $\mathcal{U}$ is not preceded by a path quantifier.

CTL presented

- Intuitive meanings:
  - $\forall \bigcirc p$: in any next state p holds.

    Regardless of the actions of the "environment", at the next clock tick p holds.
  - $\forall \Box p$: p will perpetually hold in any continuation from the current state.

    Whatever the environment does, p will hold forever.
  - $\forall p \,\mathcal{U}\, q$: in any continuation from the current state q eventually holds, and until then p must hold.

CTL formulas

- Derived operators:

$$\exists\bigcirc\phi = \neg\forall\bigcirc\phi$$
$$\forall\Diamond\phi = \forall(\text{true}\,\mathcal{U}\,\phi)$$
$$\exists\Box\phi = \neg\forall\Diamond\neg\phi$$
$$\exists\Diamond\phi = \neg\forall\Box\phi$$
$$\exists(\phi\,\mathcal{U}\,\psi) = \neg\forall(\neg\phi\,\mathcal{U}(\neg\phi\wedge\neg\psi))\wedge\neg\forall\Box\psi$$

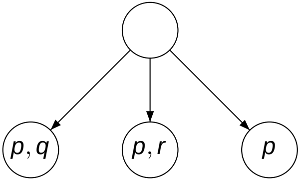
- Some intuitive meanings:
  - $\exists\bigcirc p$: there exists a next state in which p holds.

    The environment could make it possible for p to hold at the next clock tick.
  - $\exists\Box p$: there exists a continuation on which p holds perpetually.
  - $\forall\Diamond p$: in all continuations p eventually holds.

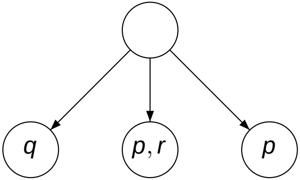
    There is a guarantee that p must eventually hold, whatever the environment does.

Branching time

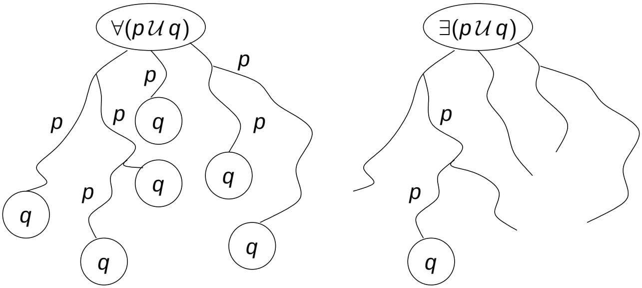
The root in the following tree satisfies $\forall \bigcirc p$:



The root in the following tree satisfies $\exists \bigcirc p$:



Branching time, contd.



Transition systems

$\mathcal{T} = (Q, \Pi, \delta, \pi, q_0)$ with

- Q finite set of *states*.
- Π finite set of *atomic propositions*.
- $q_0 \in Q$ initial state.
- $\delta \subseteq Q \times Q$ *transition relation*.
- $\pi : Q \to 2^\Pi$ *state labeling*.

Example: the hunter/wolf/goat/cabbage puzzle.

- Nondeterminism: given $q \in Q$, there may exist several $r_1, r_2, \ldots \in Q$ with $(q, r_1) \in \delta, (q, r_2) \in \delta \ldots$.
- Who chooses wich successor in each state?
  - CTL answer: the environment does!

CTL semantics in transition systems

Recursively interpret each CTL formula in each state of the system

Given $\mathcal{T} = (Q, \Pi, \delta, \pi, q_0)$ and $q \in Q$:

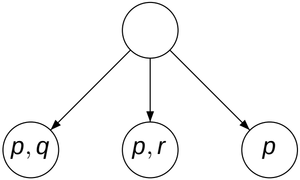
- $q \models p$ if $p \in \pi(q)$.
- $q \models \phi_1 \wedge \phi_2$ if....
- $q \models \neg\phi$ if...
- $q \models \forall\bigcirc\phi$ if for all $r \in Q$ with $(q, r) \in \delta$, $r \models \phi$. Example:

CTL semantics in transition systems

Recursively interpret each CTL formula in each state of the system

Given $\mathcal{T} = (Q, \Pi, \delta, \pi, q_0)$ and $q \in Q$:

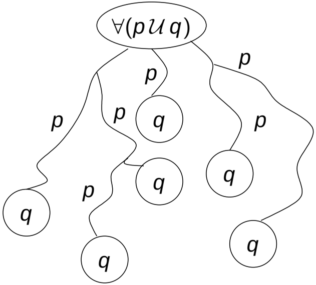
- $q \models p$ if $p \in \pi(q)$.
- $q \models \phi_1 \wedge \phi_2$ if....
- $q \models \neg\phi$ if...
- $q \models \forall\bigcirc\phi$ if for all $r \in Q$ with $(q, r) \in \delta$, $r \models \phi$. Example:



CTL semantics in transition systems (contd.)

Given $\mathcal{T} = (Q, \Pi, \delta, \pi, q_0)$ and $q \in Q$:

- $q \models \forall\Box\,\phi$ if for each run ρ in $\mathcal{T}$ starting in q with $\rho = q = q_0 \to q_1 \to \ldots \to q_n \to \ldots$ (infinite!) we have that $q_n \models \phi$ for all n.
  - In other words, $\rho \models \Box\,\phi$!
- $q \models \forall(\phi_1 \mathcal{U} \phi_2)$ if for each run ρ in $\mathcal{T}$ starting in q with $\rho = q = q_0 \to q_1 \to \ldots \to q_n \to \ldots$ there exists $n \geq 0$ with $q_n \models \phi_2$ and for all $0 \leq m < n$, $q_m \models \phi_1$.
  - In other words, $\rho \models \phi_1\,\mathcal{U}\,\phi_2$!



Property specification

CNIL

Tout utilisateur peut demander le retrait de ses données...

- How do we interpret peut?
  - p = demander le retrait... : $\forall\Box\,\exists\Diamond\, p$.

Strategy to win a game

Black has a strategy to put the game in a situation from which White king will never get close to Black pawn.

- q = White king never gets close to Black pawn : $\exists\Diamond\,\forall\Box\, q$.

Other properties related with choices, like noninterference.

CTL properties on transition systems

- Hunter/wolf/goat/cabbage puzzle.
  - Does the initial state satisfy $\forall\Diamond(h = 1 \wedge w = 1 \wedge g = 1 \wedge c = 1)$?
  - What is the right property that says that the puzzle has a solution?
- Deadlock freedom:
  - Suppose the states of each process are p_1, p_2, p_3, resp. q_1, q_2, q_3.
  - Deadlock freedom, i.e. all computations may progress:

$$\forall\Box \bigvee_{1\leq i\leq 3} (PC_1 = p_i \wedge \exists\bigcirc PC_1 \neq p_i) \vee \bigvee_{1\leq i\leq 3} (PC_2 = q_i \wedge \exists\bigcirc PC_2 \neq q_i)$$

CTL properties on transition systems

- Hunter/wolf/goat/cabbage puzzle.
  - Does the initial state satisfy $\forall\Diamond(h = 1 \wedge w = 1 \wedge g = 1 \wedge c = 1)$?
  - What is the right property that says that the puzzle has a solution?
    $\exists\Diamond(h = 1 \wedge w = 1 \wedge g = 1 \wedge c = 1)$
- Deadlock freedom:
  - Suppose the states of each process are p_1, p_2, p_3, resp. q_1, q_2, q_3.
  - Deadlock freedom, i.e. all computations may progress:

$$\forall\Box \bigvee_{1\leq i\leq 3} (PC_1 = p_i \wedge \exists\bigcirc PC_1 \neq p_i) \vee \bigvee_{1\leq i\leq 3} (PC_2 = q_i \wedge \exists\bigcirc PC_2 \neq q_i)$$

CTL properties on transition systems

- Hunter/wolf/goat/cabbage puzzle.
  - Does the initial state satisfy $\forall\Diamond(h = 1 \wedge w = 1 \wedge g = 1 \wedge c = 1)$?
  - What is the right property that says that the puzzle has a solution? $\exists\Diamond(h = 1 \wedge w = 1 \wedge g = 1 \wedge c = 1)$
- Deadlock freedom:
  - Suppose the states of each process are p_1, p_2, p_3, resp. q_1, q_2, q_3.
  - Deadlock freedom, i.e. all computations may progress:

$$\forall\Box \bigvee_{1 \leq i \leq 3} (PC_1 = p_i \wedge \exists\bigcirc PC_1 \neq p_i) \vee \bigvee_{1 \leq i \leq 3} (PC_2 = q_i \wedge \exists\bigcirc PC_2 \neq q_i)$$

Sample tautologies

- Tautology : formula that is true regardless of the truth values given to the atomic propositions.
- Examples:

$$\neg\forall\bigcirc p \leftrightarrow \exists\bigcirc\neg p$$
$$\forall\bigcirc p \rightarrow \forall\Diamond p$$
$$\exists\Diamond\exists\Diamond p \rightarrow \exists\Diamond p$$
$$\forall\Box(p\wedge q) \leftrightarrow \forall\Box p \wedge \forall\Box q$$
$$(\exists\Diamond p \rightarrow \exists\Diamond q) \rightarrow \exists\Diamond(p \rightarrow q)$$

- Formulas which are not tautologies:

$$\forall\Diamond(p\vee q) \leftrightarrow \forall\Diamond p \vee \forall\Diamond q$$

- To prove they are not tautologies, give a counter-model!

Minimal set of operators

All CTL formulas can be expressed using the following set of operators :

- Boolean operators (further reducible, e.g., to $\wedge$ and $\neg$).
- $\forall \bigcirc$.
- $\forall \mathcal{U}$.
- $\forall \Box$.

Examples – express the following:

- $\exists (p \,\mathcal{U}\, q)$.
- $\exists \Box p$.

The dual set of path-temporal operators can also be used as minimal set of operators!

Other (linear) temporal operators: weak until, release

- Weak until $p \mathcal{W} q$: $p \mathcal{W} q \equiv p \mathcal{U} q \wedge \Box p$.
- Release $p \mathcal{R} q$: $p \mathcal{R} q \equiv \neg(\neg p \mathcal{U} \neg q)$.
- Can be extended to CTL operators: $\forall p \mathcal{W} q$, $\exists p \mathcal{R} q$, etc.

Fixpoints

- Globally, forward, until, release can be defined "inductively":

$$\exists \Diamond p \equiv p \vee \exists \bigcirc \exists \Diamond p$$
$$\forall \Diamond p \equiv \ldots ?$$
$$\exists \Box p \equiv \ldots ?$$
$$\forall \Box p \equiv \ldots ?$$
$$\exists p \,\mathcal{U}\, q \equiv q \vee \big(p \wedge \exists \bigcirc (p \,\mathcal{U}\, q)\big)$$
$$\forall p \,\mathcal{U}\, q \equiv \ldots ?$$
$$\exists p \,\mathcal{R}\, q \equiv q \wedge \big(p \vee \bigcirc \exists (p \,\mathcal{R}\, q)\big)$$

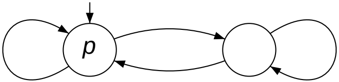
Remarks on LTL vs. CTL (to be continued!)

- Both LTL and CTL formulas are interpreted over transition systems.
- An LTL formula speaks about what happens on one run that starts in a state.
  - Time passage is determined by some superior entity, choices do not exist and no dilemma about possible continuations exists.
  - A posteriori analysis of the behavior of a system (but behaviors may be infinite!).
- A CTL formula speaks about what could happen in various runs that starts in a state.
  - Time is nondeterministic and choices must be taken into account, good/bad things may happen due to good/bad decisions and continuations depend on them.
  - A priori analysis of the possible evolution of a system.
- Some LTL formulas (but not all!) can be represented as CTL formulas:
  - Checking $\Box p$ holds at a state q in a transition system requires checking that all runs starting in q satisfy $\Box p$.
  - Hence, from this state-centered point of view, checking $\Box p$ amounts to checking $\forall\Box p$.
  - No longer holds for more complex formulas!
  - Simply because $\forall(\Diamond p \wedge \Box q)$ is not a CTL formula!

The model-checking problem

- Given a CTL formula ϕ and a finitely presentable model M, does $M \models \phi$ hold?
  - Finitely presentable tree = transition system over AP.
  - The tree = the unfolding of $\mathcal{A}$.
- Note the difference with LTL models :
  - A transition system embodies an uncountable set of models for LTL !
  - A transition system embodies a unique model for CTL !

CTL model-checking instances



- Which state satisfies $\exists\Diamond p$?
  - Search for a reachable state labeled with p.
- Which state satisfies $\exists\Box p$?
  - Search for a reachable strongly connected set labeled with p.
  - Only states in this SCC satisfy $\exists\Box p$.

CTL model-checking [Clarke & Emerson]

- State labeling algorithm:
  - Given formula ϕ, **split** Q into Q_ϕ and $Q_{\neg\phi}$
  - Structural induction on the syntactic tree of ϕ.
  - Add a new propositional symbol p_ϕ for each analyzed ϕ.
  - Label Q_ϕ with p_ϕ and do not label $Q_{\neg\phi}$ with p_ϕ.

CTL model-checking (2)

- For $\phi = \forall\bigcirc p$

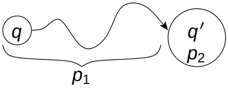
$$Q_{\forall\bigcirc p} = \{q \in Q \mid \forall q' \in \delta(q), p \in \pi(q')\}$$
$$Q_{\neg\forall\bigcirc p} = \{q \in Q \mid \exists q' \in \delta(q), p \notin \pi(q')\}$$

- Example...

CTL model-checking (3)

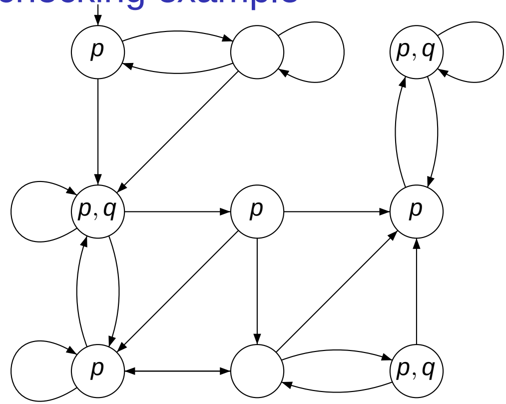
- $\phi = \exists \Box p$.
  - $Q_{\exists \Box p}$ contains state q iff q is labeled with p and belongs to a circuit containing only p states.
  - $Q_{\neg \exists \Box p} = Q \setminus Q_{\exists \Box p}$.
- Example...

CTL model-checking (4)

- $\phi = \exists(p_1 \,\mathcal{U}\, p_2)$
  - $Q_{\exists(p_1 \,\mathcal{U}\, p_2)}$ contains state q iff $\exists q' \in Q$ s.t.:
    - 

  - $Q_{\neg\exists(p_1 \,\mathcal{U}\, p_2))} = Q \setminus Q_{\exists(p_1 \,\mathcal{U}\, p_2)}$.
- Example...

CTL model-checking example



$\exists \Box p$ $\qquad$ $\forall \Box p$

$\exists p \,\mathcal{U}\, q$ $\qquad$ $\forall p \,\mathcal{U}\, q$ $\qquad$ $\exists \bigcirc \forall \bigcirc p$

Properties of the (first variant of the) model-checking algorithm

- It seems that the model-checking algorithm requires graph algorithms
  - Successors for $\exists \bigcirc$.
  - Reachability analysis for $\exists \mathcal{U}$.
  - Circuits for $\exists \Box$.
- But could we take advantage of the fixpoint expansions of the temporal operators?

$$\exists \Box p \equiv p \wedge \exists \bigcirc \exists \Box p$$
$$\exists p \,\mathcal{U}\, q \equiv q \vee \big(p \wedge \exists \bigcirc (p \,\mathcal{U}\, q)\big)$$

Fixpoint variant of the model-checking algorithm

- Given a formula ϕ and a transition system $M = (Q, q_0, \delta)$,
- ... denote $Sat_M(\phi)$ the set of states in Q which satisfy ϕ.
- ... and denote $post(q) = \{r \in Q \mid (q,r) \in \delta\}$.

Theorem

- *$Sat(\exists(\phi\,\mathcal{U}\,\psi))$ is the* smallest *subset T of Q such that:*
  1. *$Sat(\psi) \subseteq T$ and*
  2. *If $q \in Sat(\phi)$ and $post(q) \cap T \neq \emptyset$ then $q \in T$.*
- *$Sat(\forall\Box\phi)$ is the* largest *subset T of Q such that:*
  3. *$Sat(\psi) \supseteq T$ and*
  4. *If $q \in T$ then $post(q) \cap T \neq \emptyset$.*

The last line can also be read as:

4. For any $q \in Q$, if $post(q) \cap T = \emptyset$ then $q \notin T$.

Fixpoint variant of the model-checking algorithm

How to compute $Sat(\exists(\phi \mathcal{U} \psi))$:

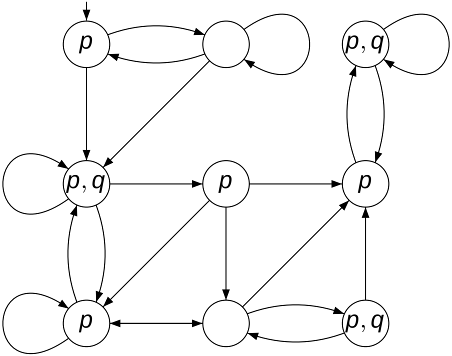
1. Start with $T = Sat(\psi)$.
2. Append q to T if $q \in Sat(\phi)$ and $post(q) \cap T \neq \emptyset$.
3. .... until T no longer grows.

How to compute $Sat(\exists \Box \phi)$:

1. Start with $T = Sat(\phi)$.
2. Eliminate, inductively, from T all states for which $post(q) \cap T = \emptyset$.
3. ... until T no longer diminishes.

Examples....

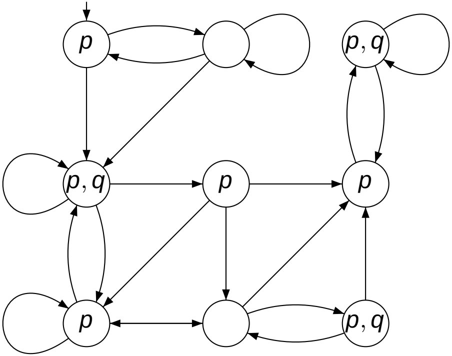
Fixpoint variant of the model-checking algorithm



$\exists(\exists \bigcirc q \,\mathcal{U}\, \forall \bigcirc p)$

- Compute $Sat(\exists \bigcirc q)$.
- Compute $Sat(\forall \bigcirc p)$.
- Instantiate $T = Sat(\forall \bigcirc p)$.
- Append st to T if $st \in Sat(\exists \bigcirc q)$ and $post(st) \in T$.

Fixpoint variant of the model-checking algorithm



$\exists(\exists \bigcirc q \,\mathcal{U}\, \forall \bigcirc p)$ $\forall(\exists \Diamond p \,\mathcal{U}\, \exists \Box q)$

post and *pre*

How to compute $Sat(\exists \phi \mathcal{U} \psi)$:

1. Start with $T = Sat(\psi)$.
2. Append q to T if $q \in Sat(\phi)$ and $post(q) \cap T \neq \emptyset$.
3. The same with $T := pre(T) \cap Sat(\phi)$.
4. Here $pre(T) = \{q \mid \exists r \in Q, (q, r) \in \delta\}$.

How to compute $Sat(\exists \Box \phi)$:

1. Start with $T = Sat(\phi)$.
2. Eliminate, inductively, from T all states for which $post(q) \cap T = \emptyset$.
3. The same with $T := \overline{pre}(T) \cap T$
4. Here $\overline{pre}(T) = Q \setminus pre(Q \setminus T)$.
5. In other words, $\overline{pre}(T)$ contains all the states whose successors all belong to T.

post and *pre*

How to compute $Sat(\exists\phi\,\mathcal{U}\,\psi)$:

1. Start with $T = Sat(\psi)$.
2. Append q to T if $q \in Sat(\phi)$ and $post(q) \cap T \neq \emptyset$.
3. The same with $T := pre(T) \cap Sat(\phi)$.
4. Here $pre(T) = \{q \mid \exists r \in Q, (q, r) \in \delta\}$.

How to compute $Sat(\exists\Box\phi)$:

1. Start with $T = Sat(\phi)$.
2. Eliminate, inductively, from T all states for which $post(q) \cap T = \emptyset$.
3. The same with $T := \overline{pre}(T) \cap T$
4. Here $\overline{pre}(T) = Q \setminus pre(Q \setminus T)$.
5. In other words, $\overline{pre}(T)$ contains all the states whose successors all belong to T.