Proof circuits and other models of parallel computation

Logic and interactions 2012

Clément Aubert
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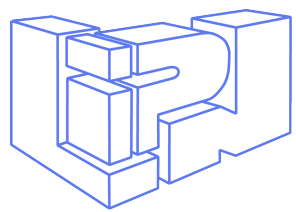

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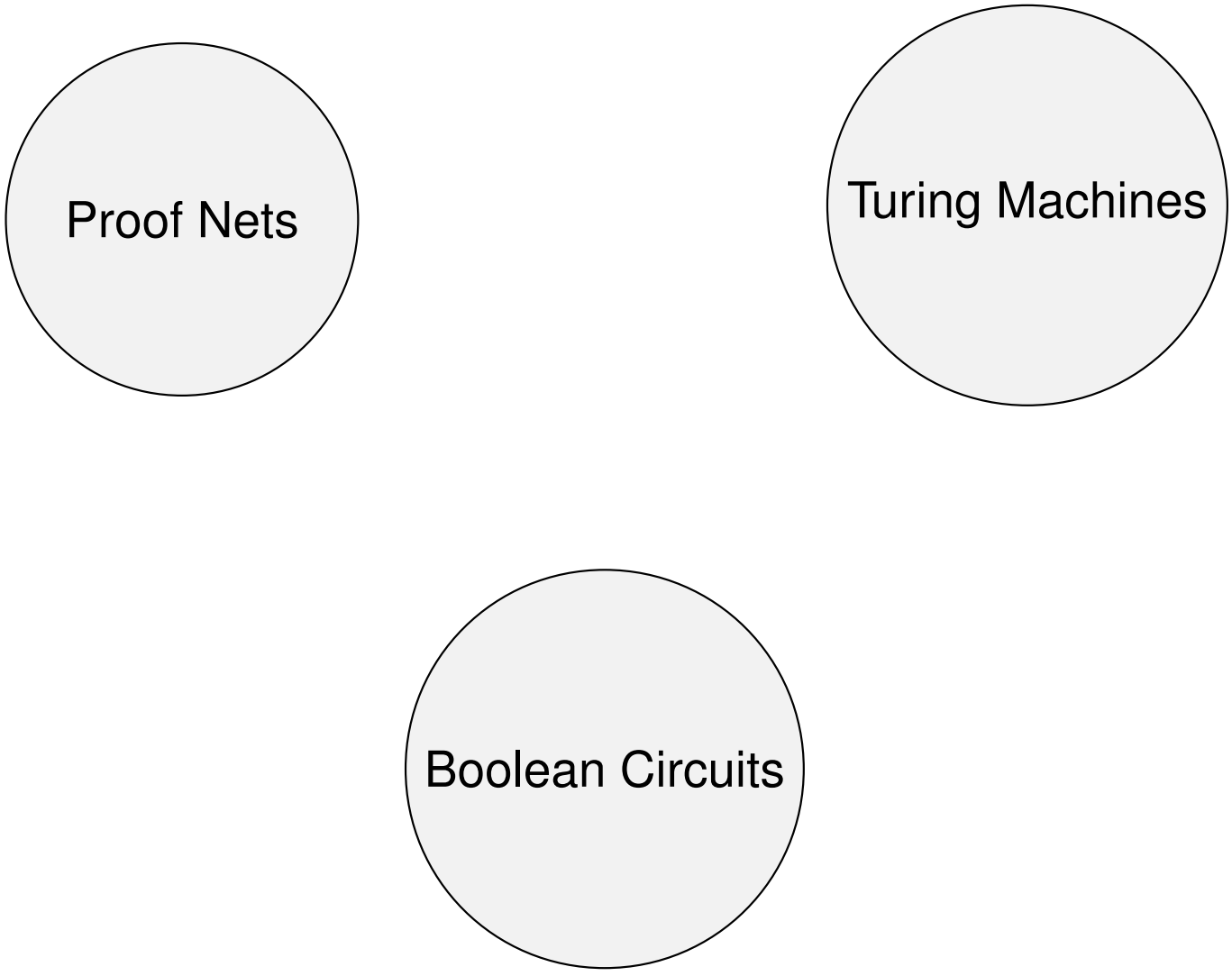
93430 Villetaneuse

Thursday 2 February 2012

The big picture



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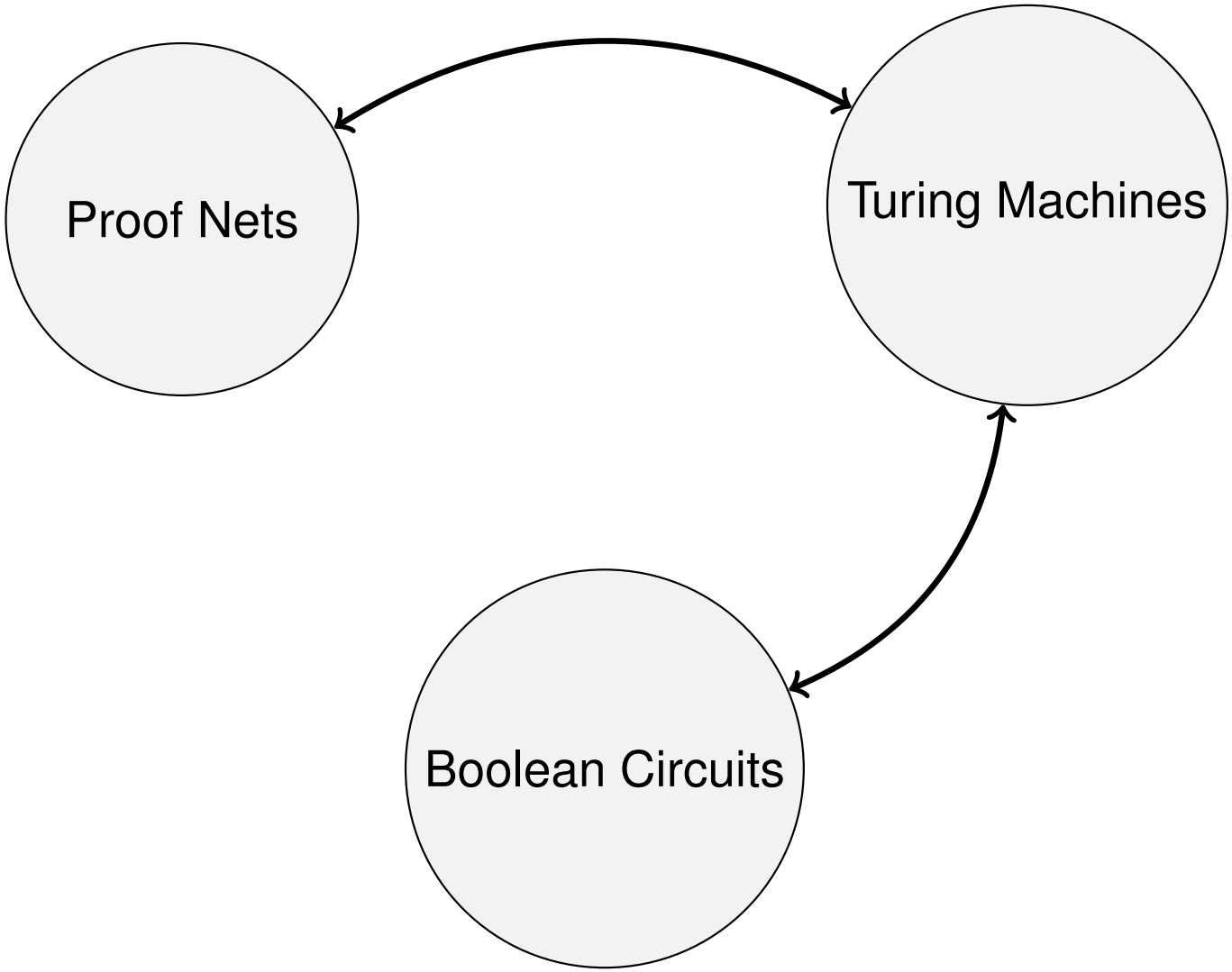

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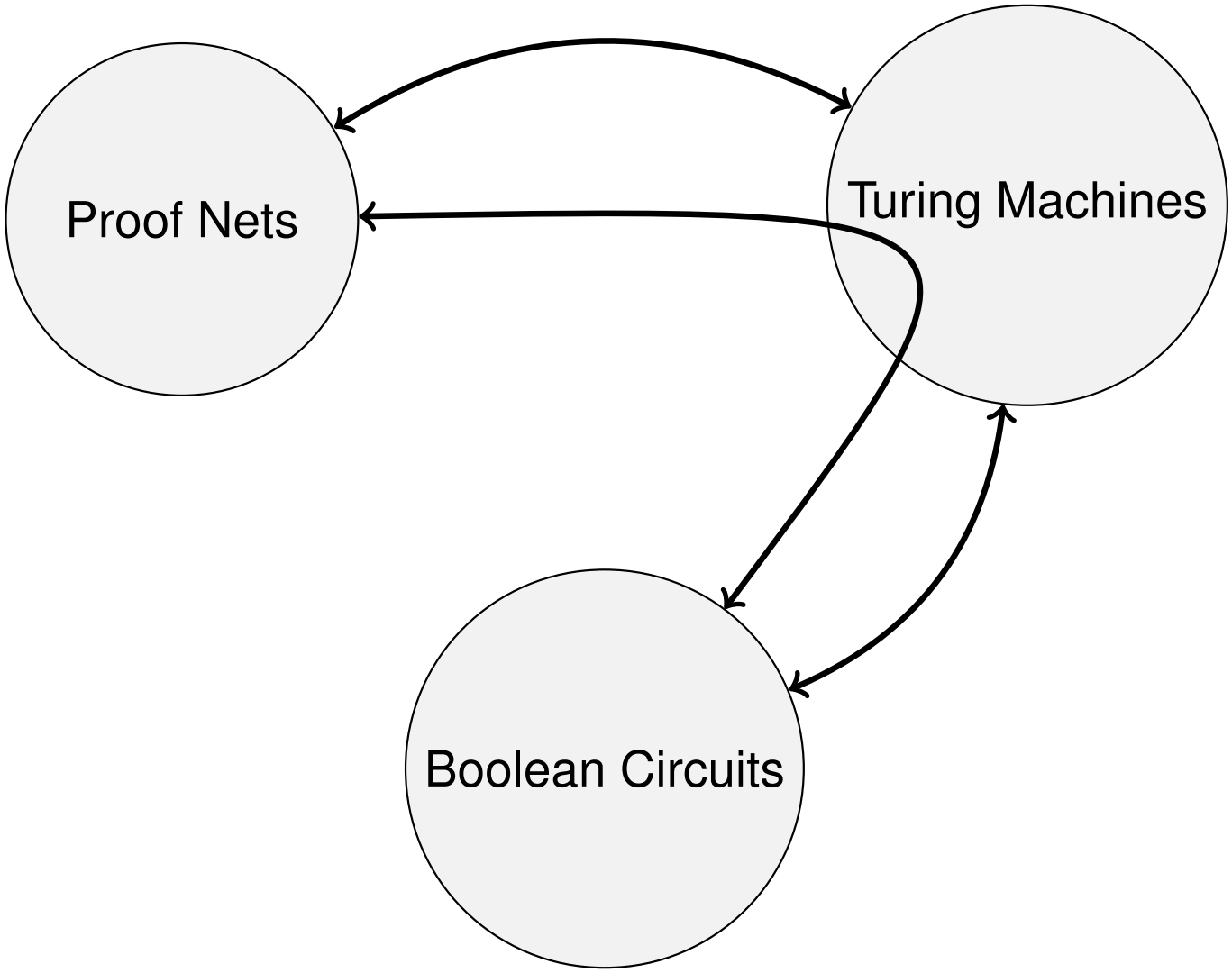

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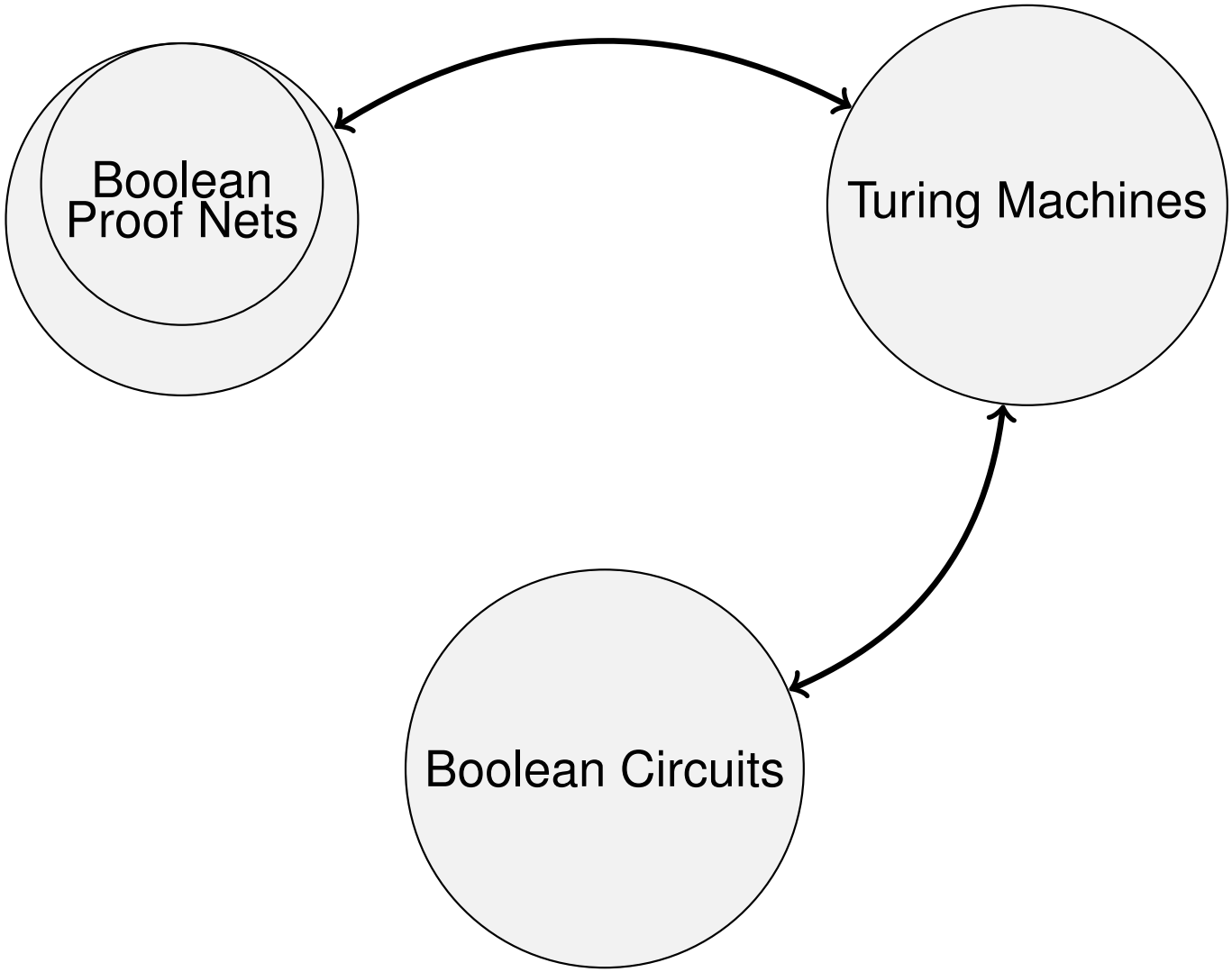

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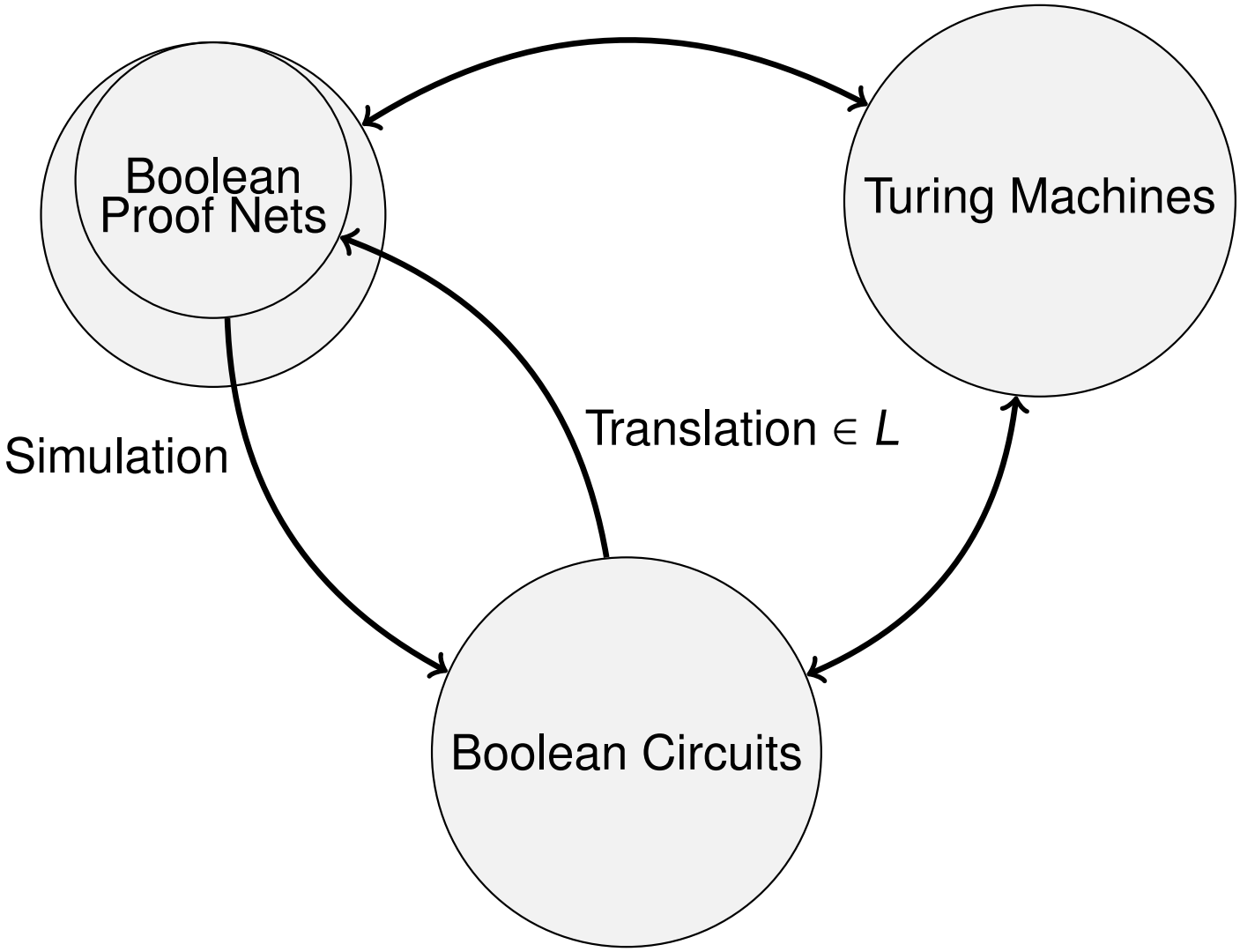

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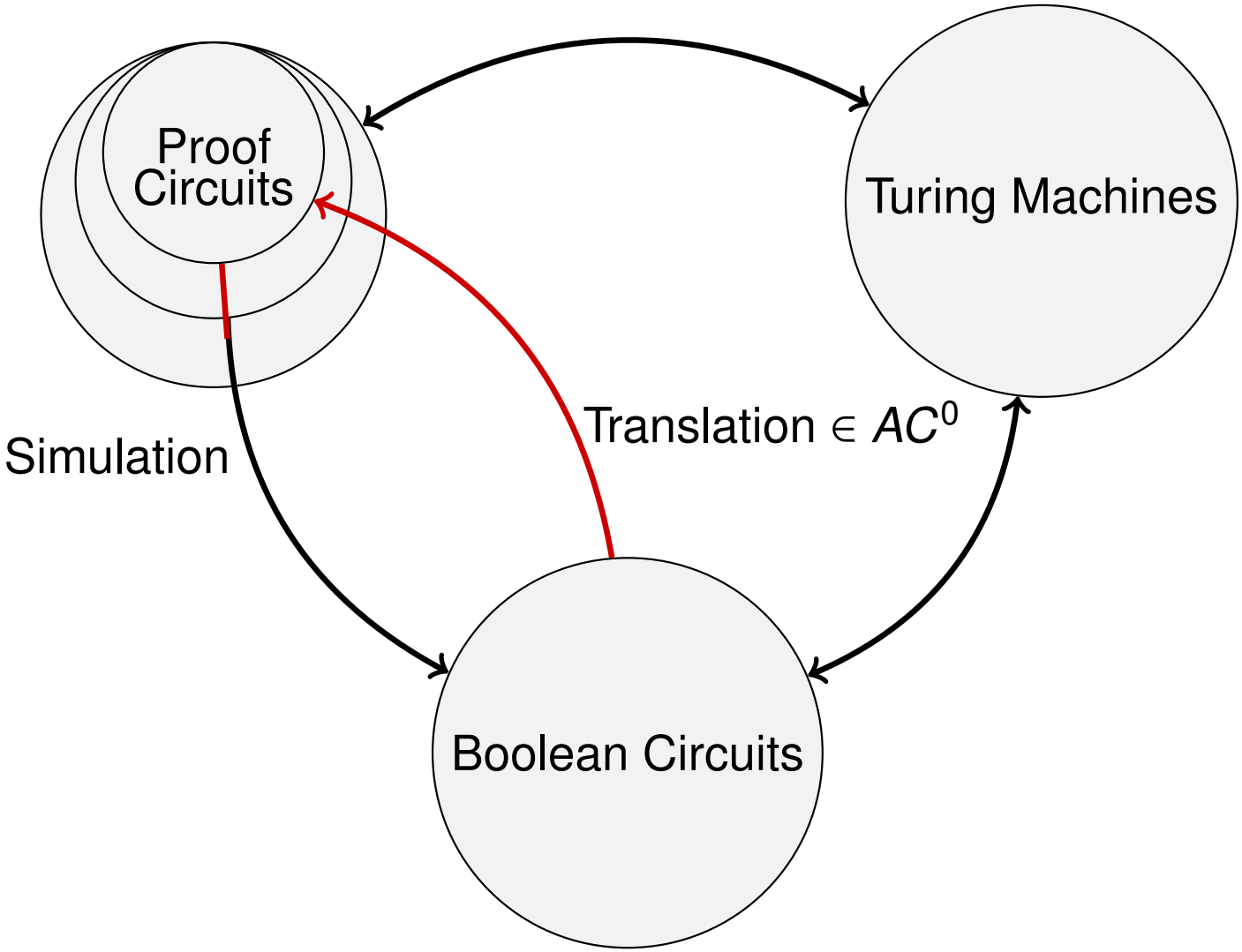

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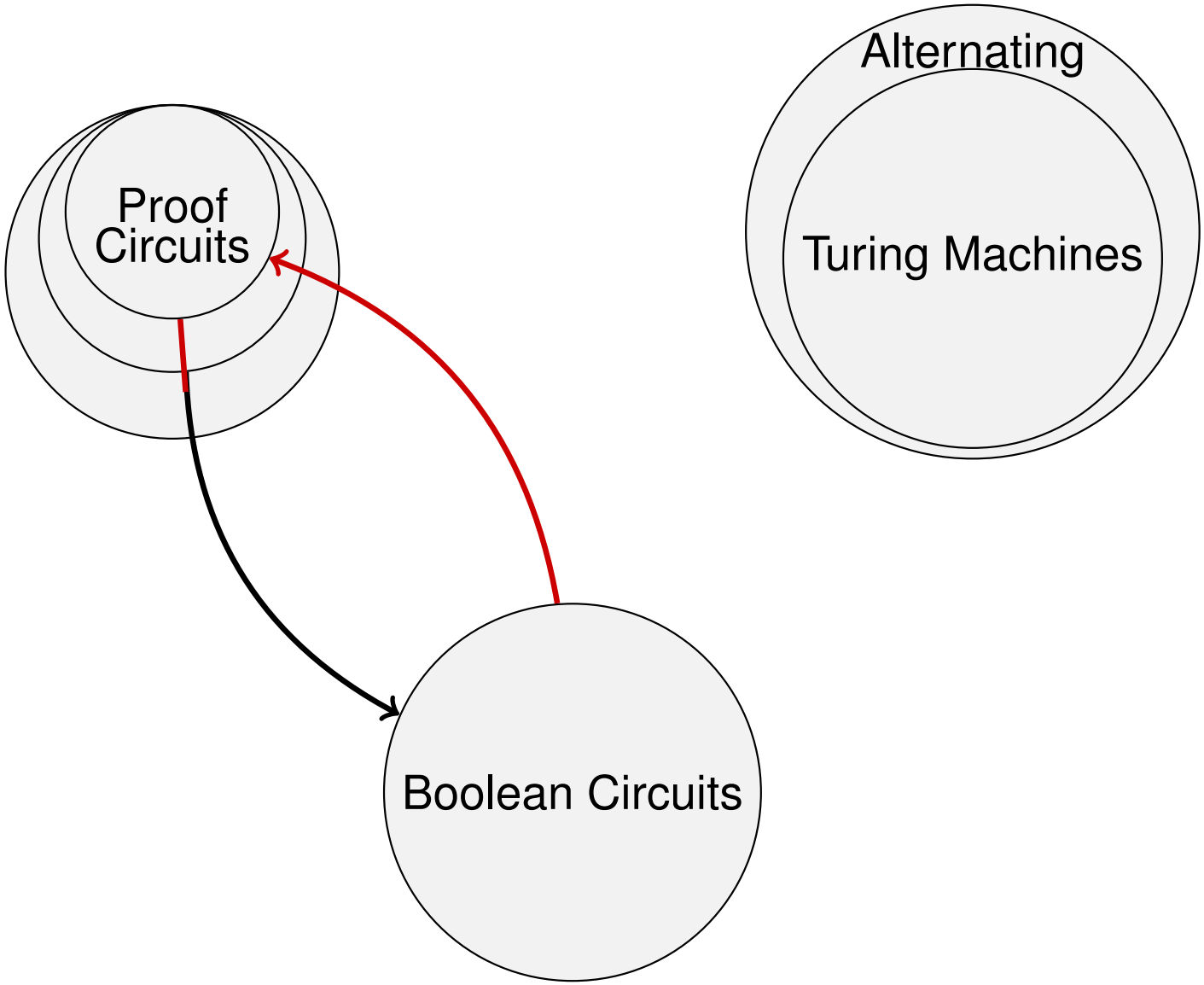

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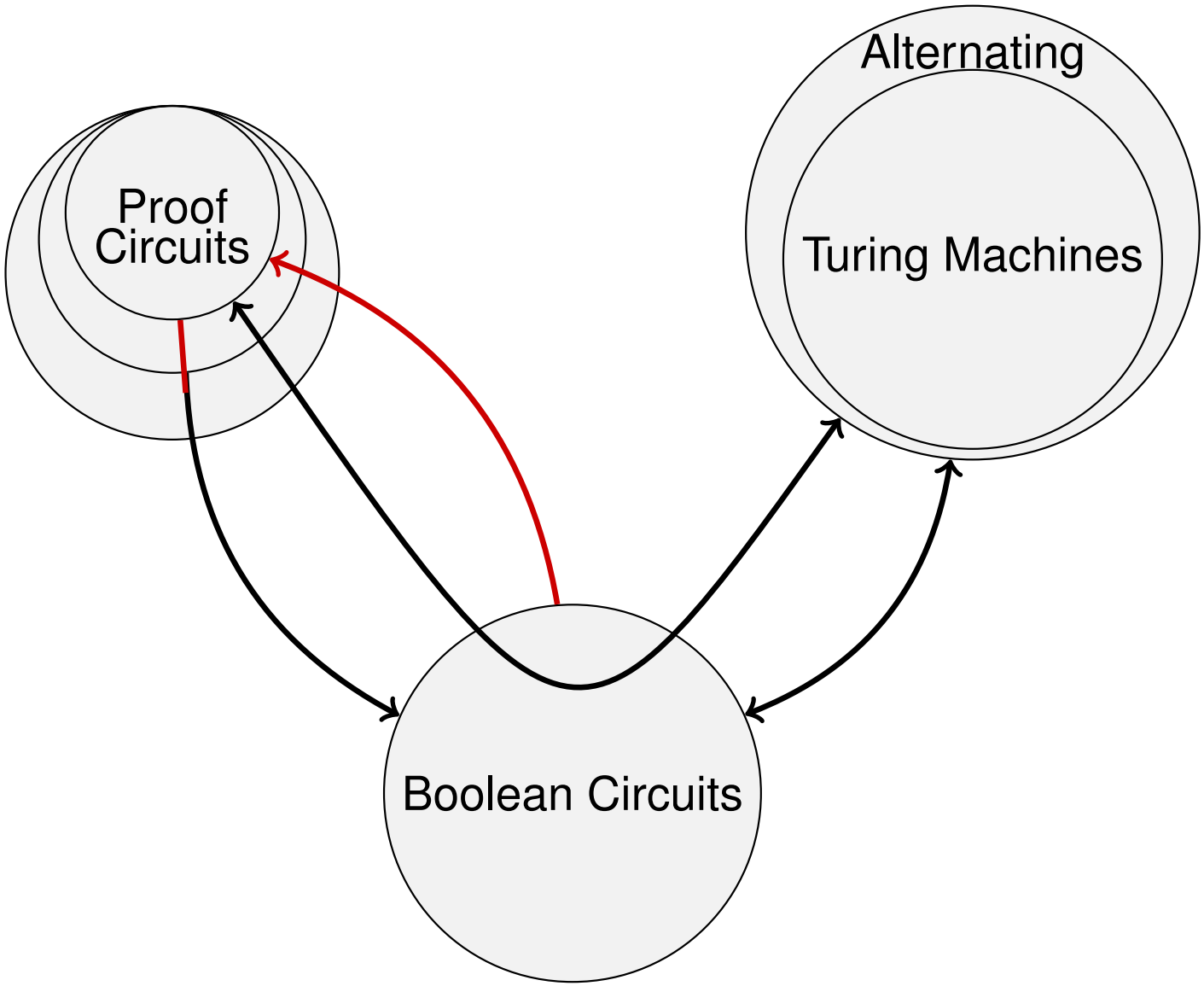

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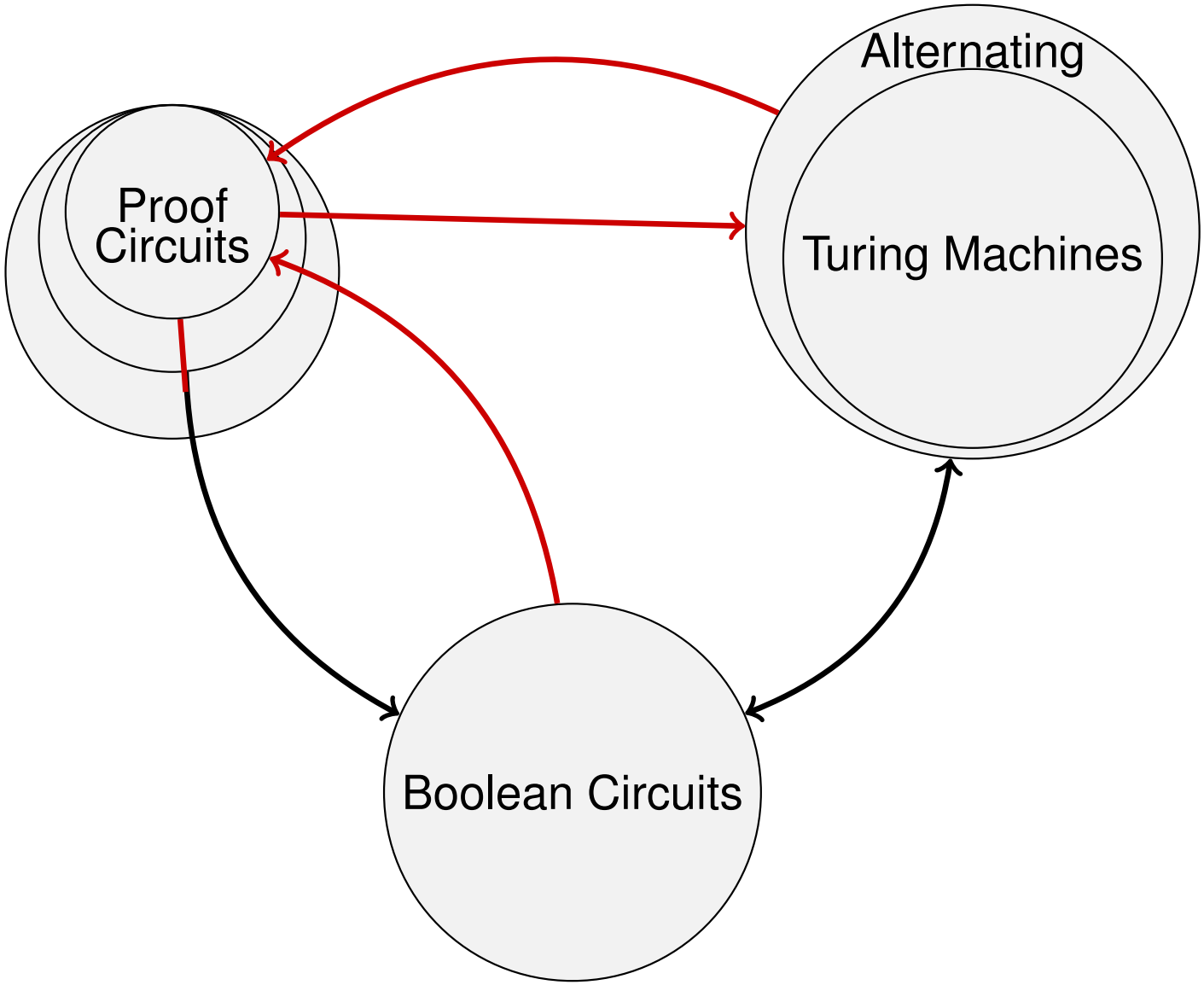
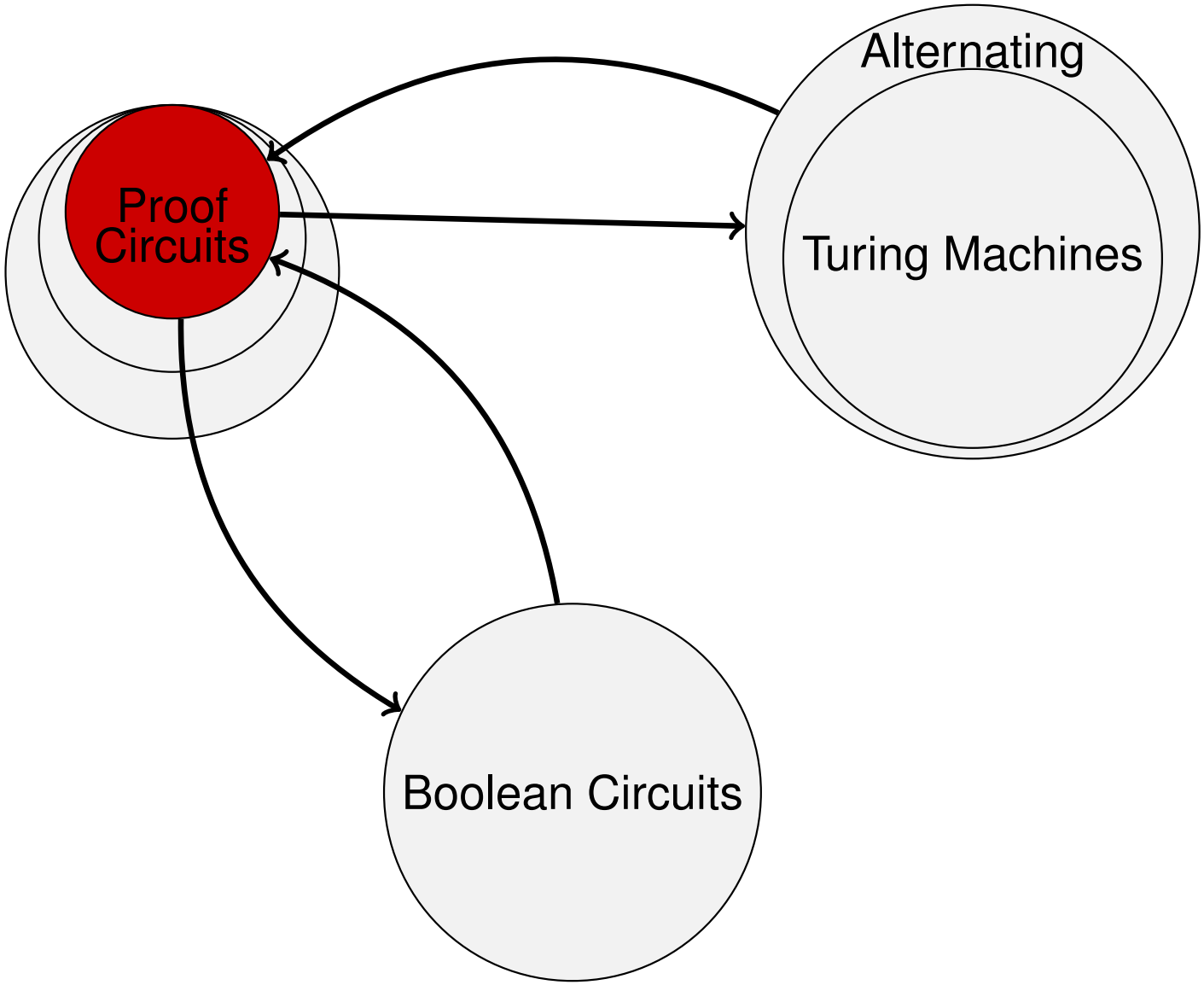
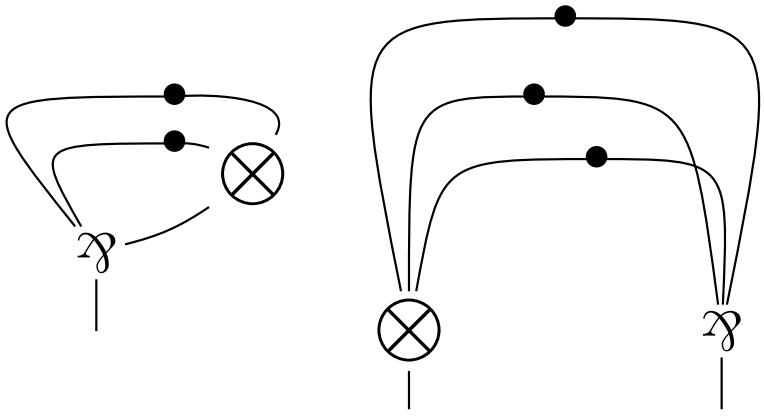


Map (1~5): Proof Circuits



How to compute with Proof nets?



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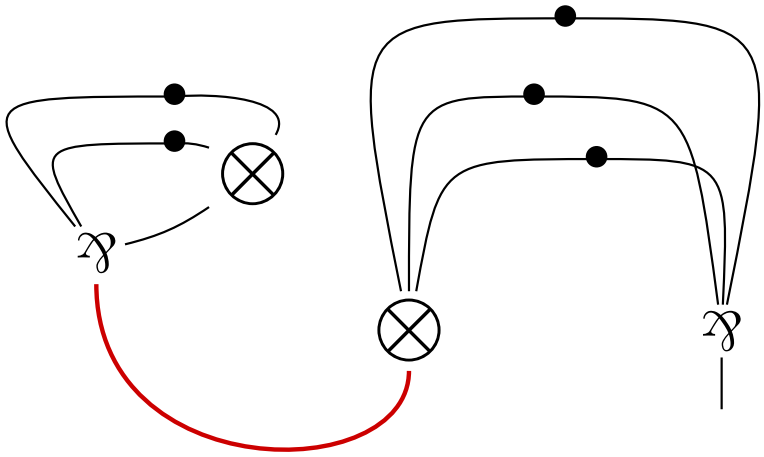

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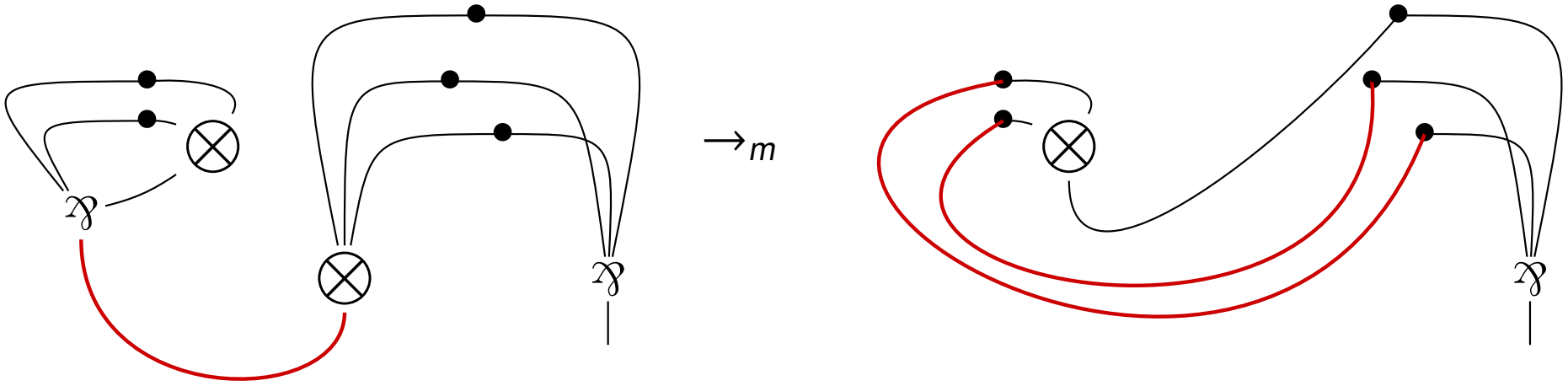

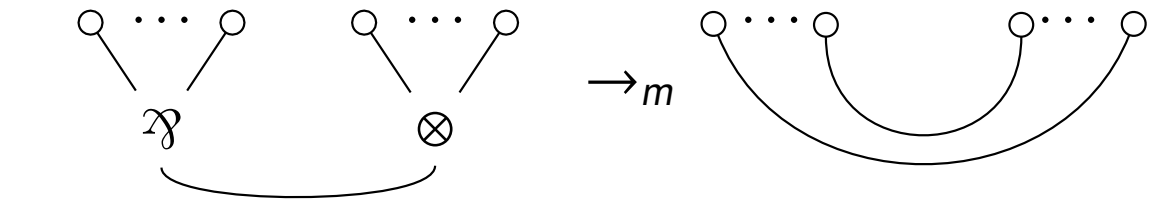

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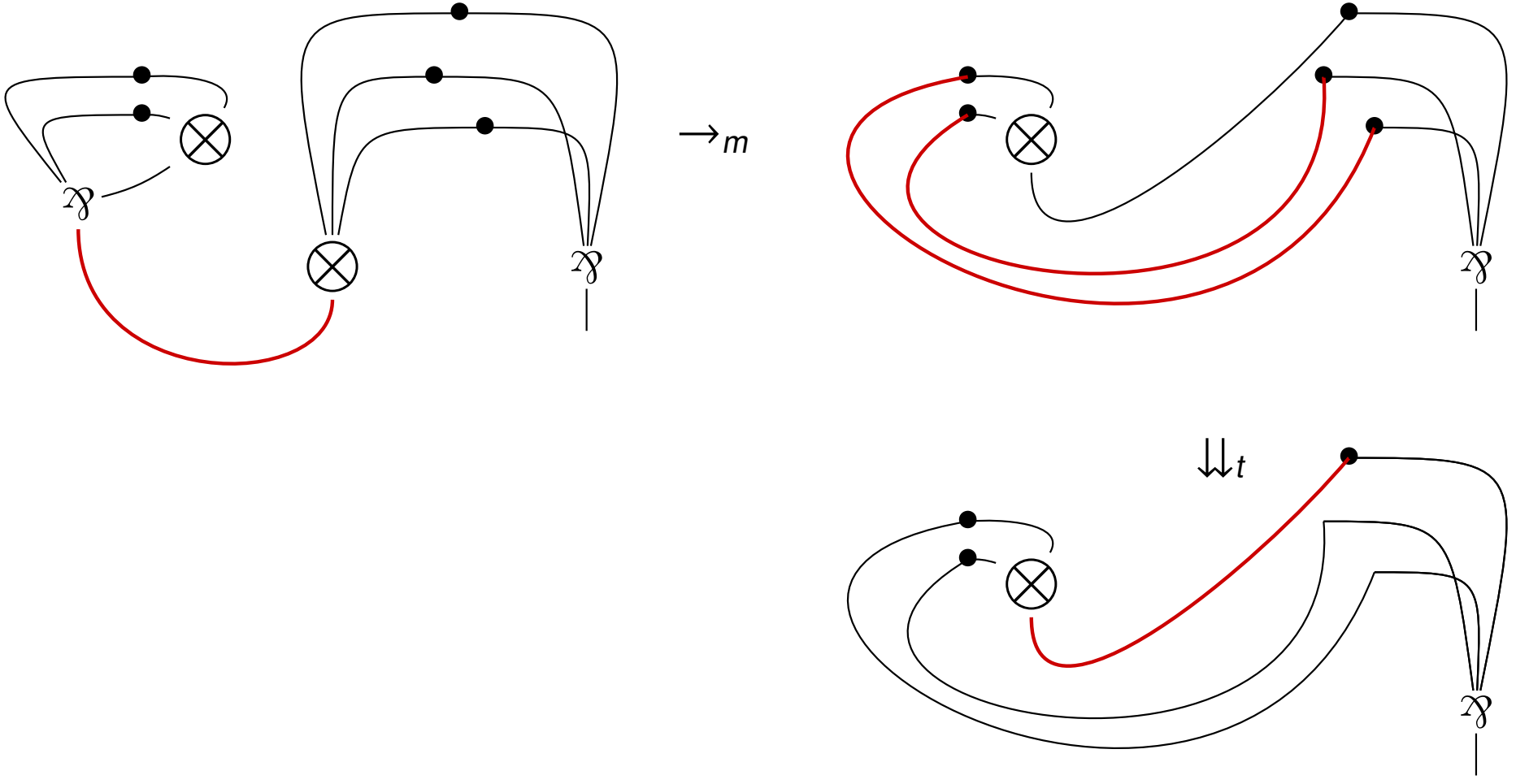

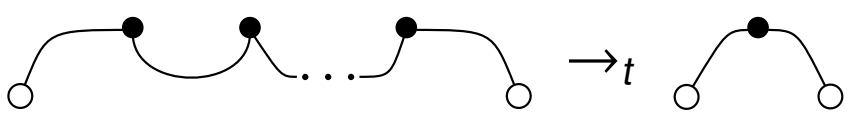

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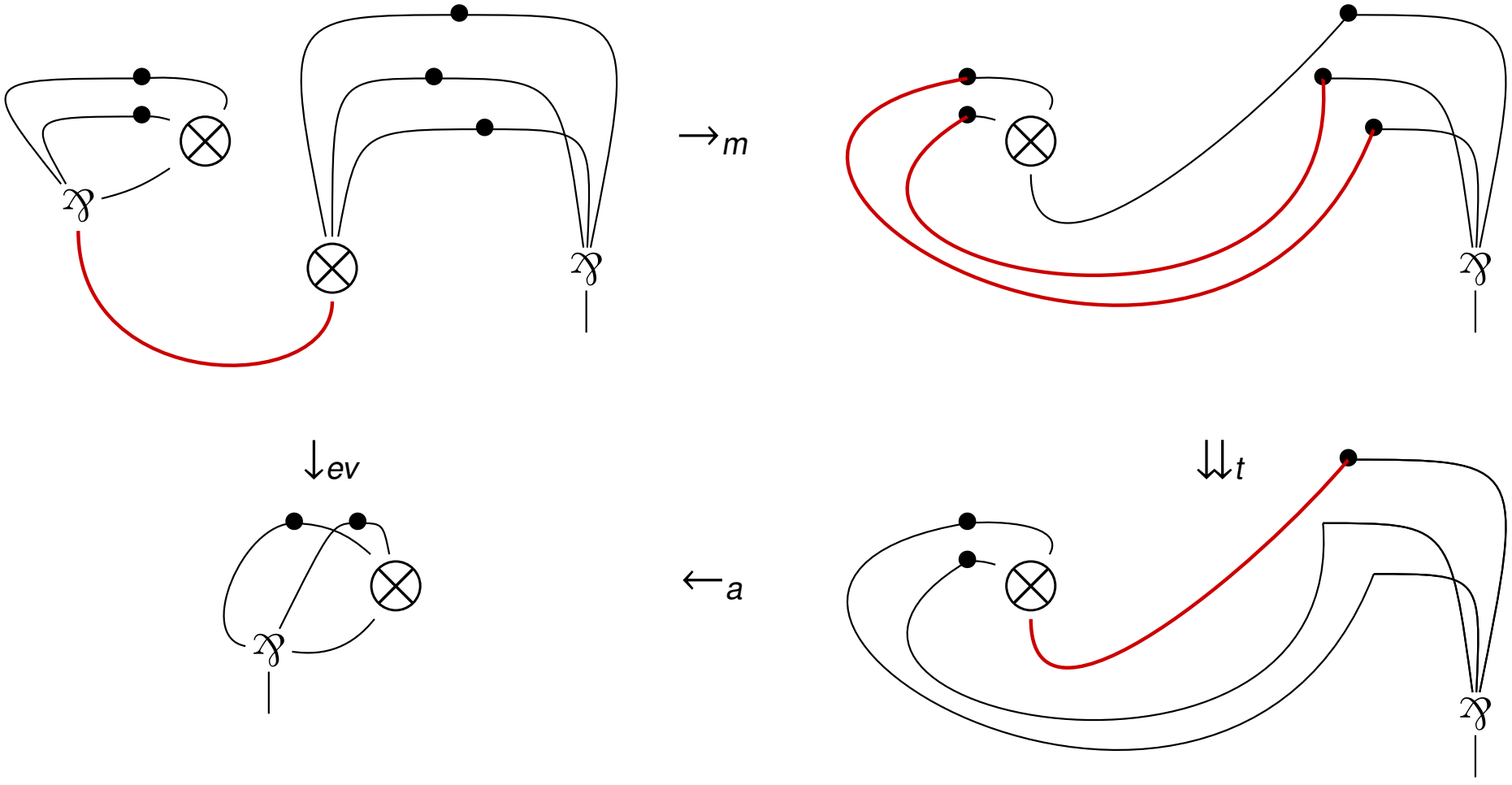
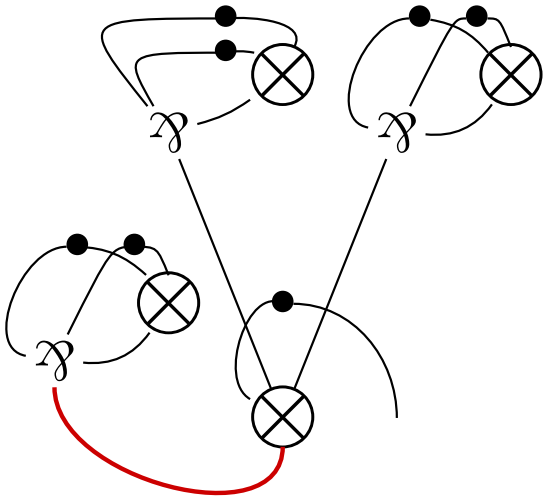


An example: the conditional



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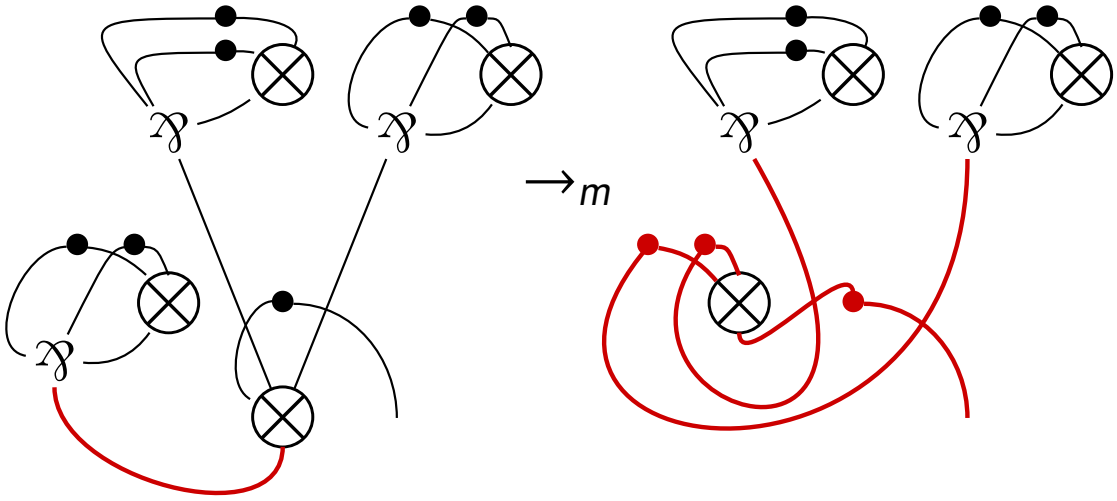

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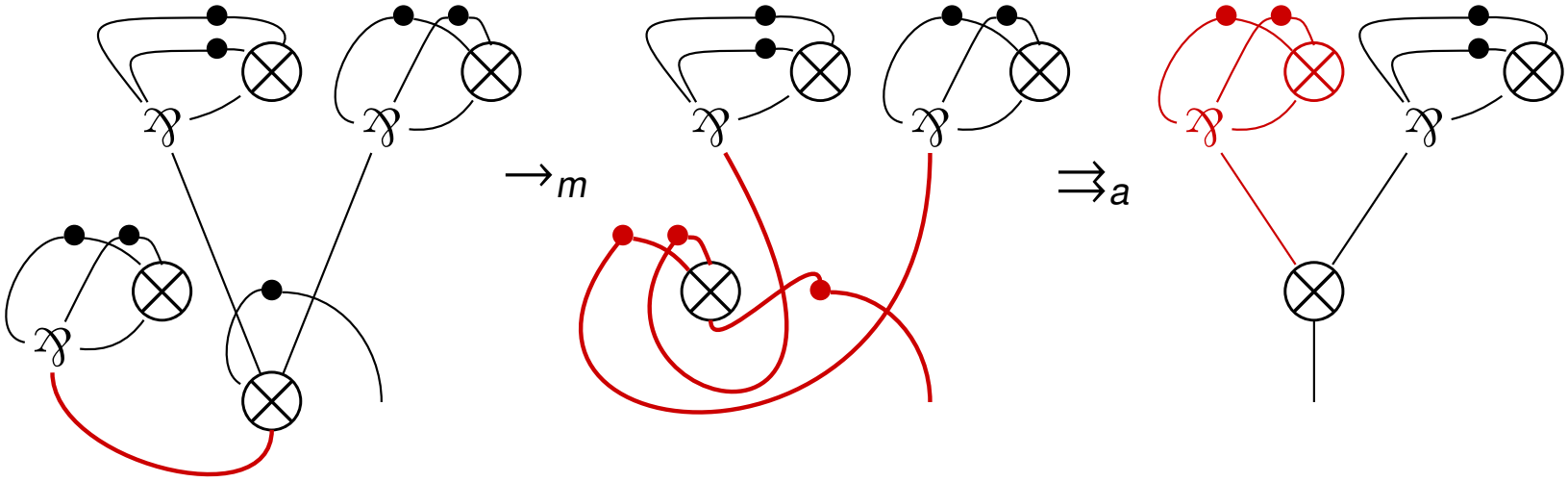
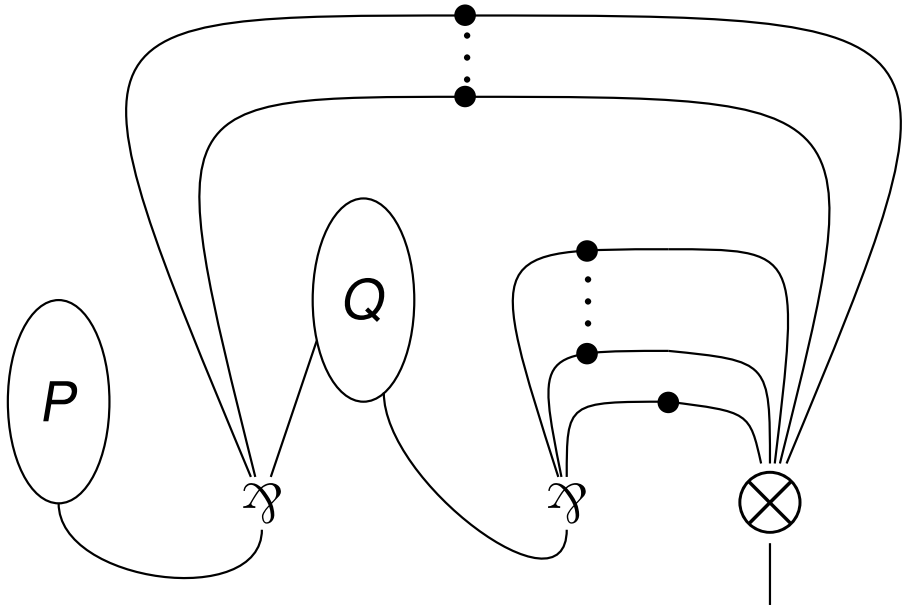
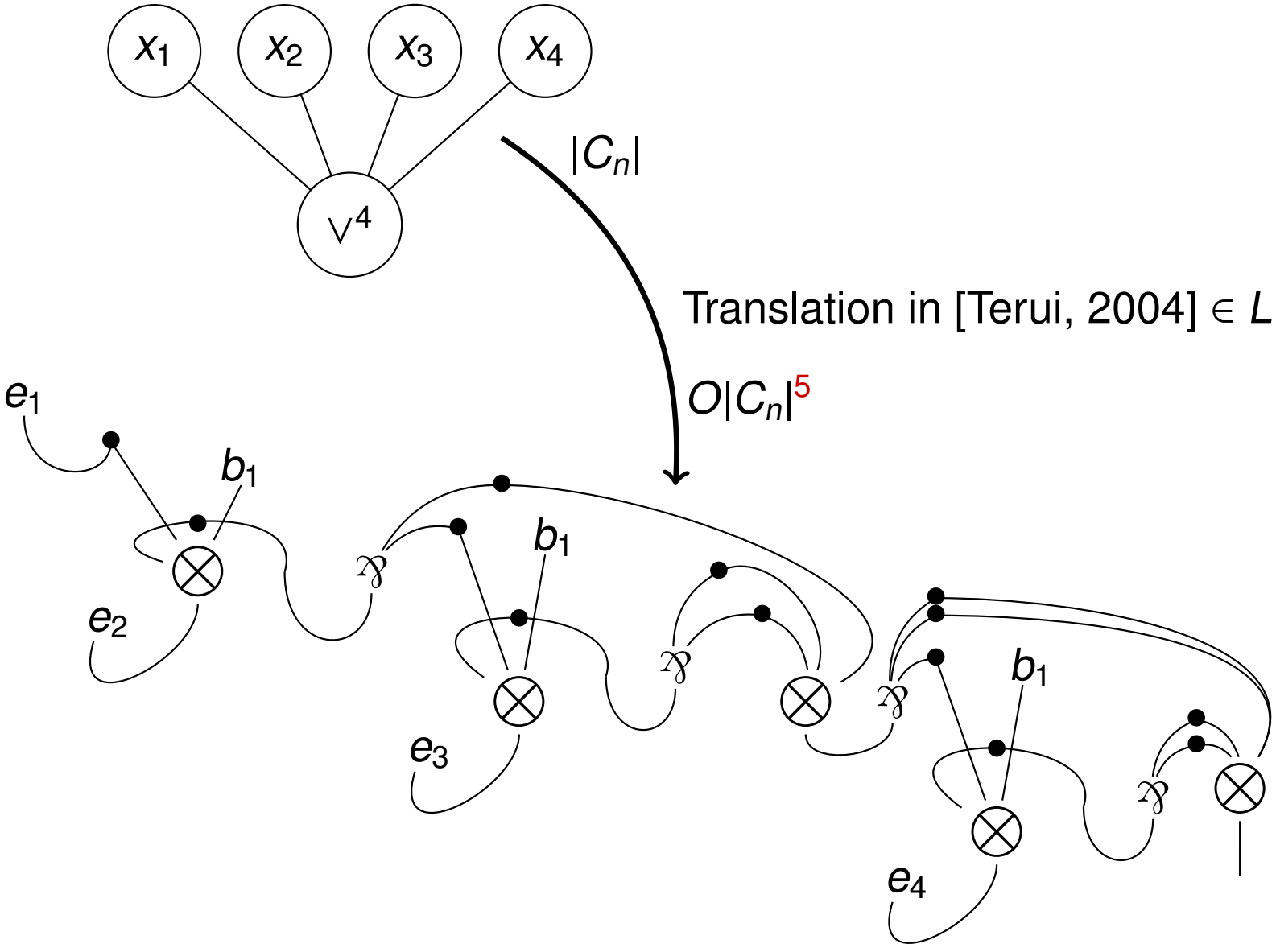
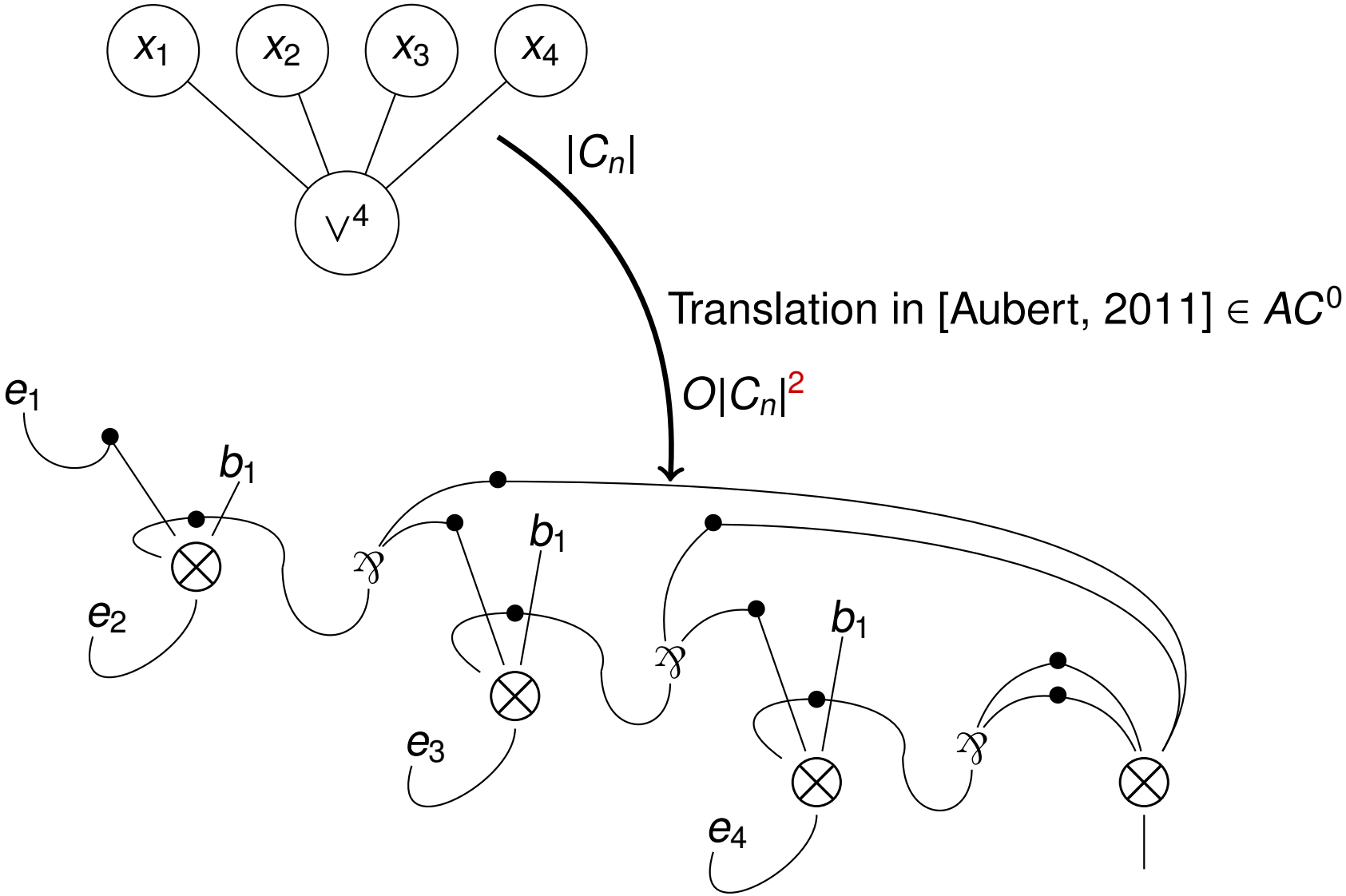




$\vee^4$ translated

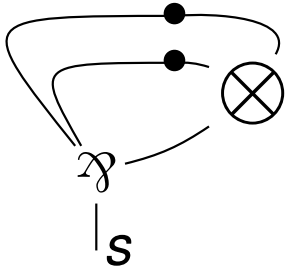


$\vee^4$ translated

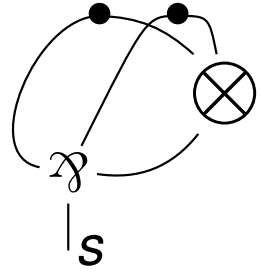


Pieces (1/4)

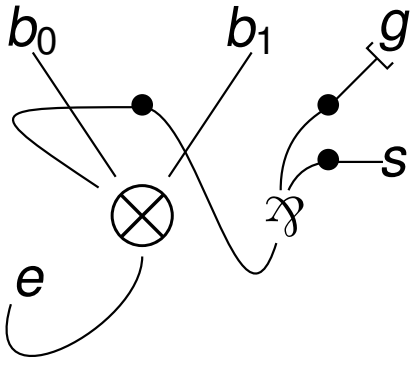
$b_0 \equiv$



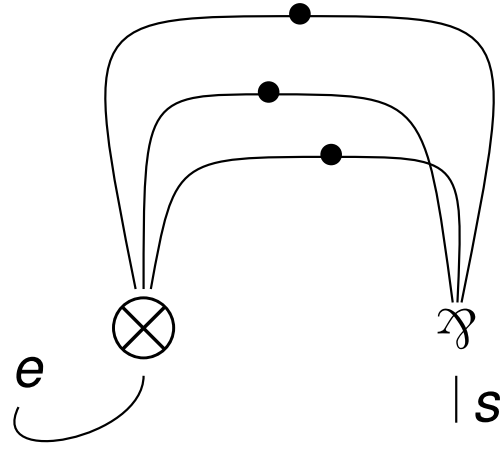
$b_1 \equiv$



COND $\equiv$

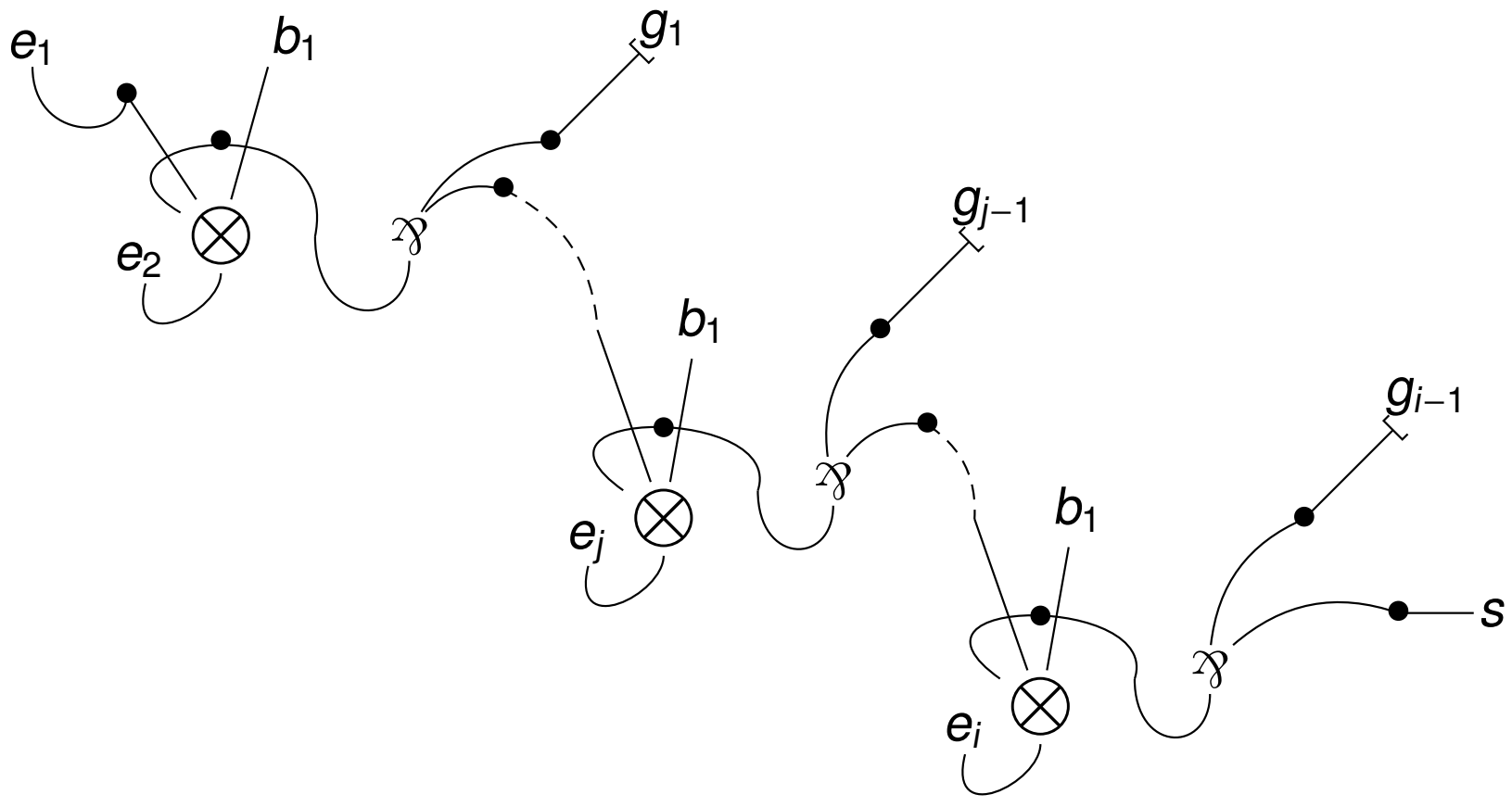


NEG $\equiv$



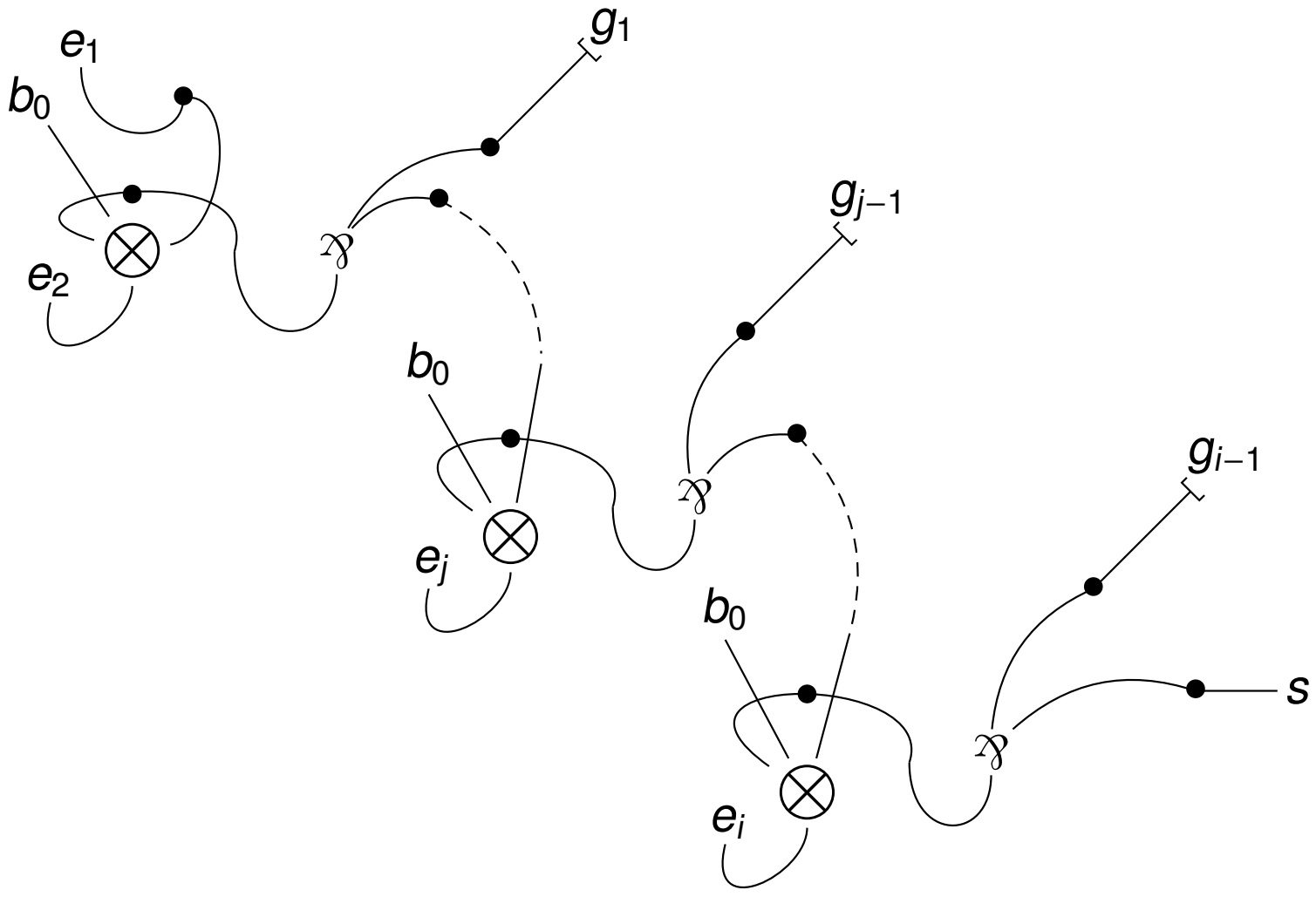
Pieces (2/4)

$DISJ^i \equiv$



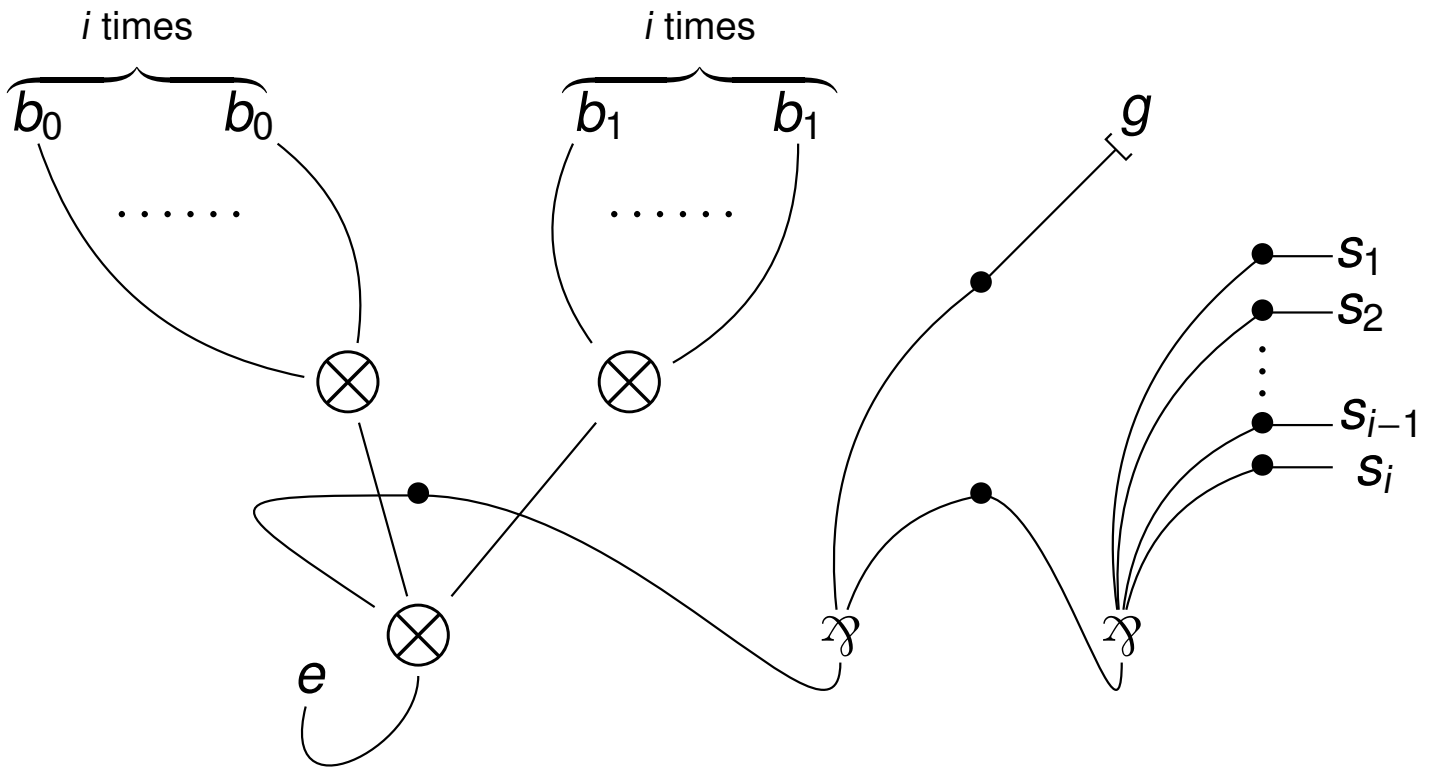
Pieces (3/4)

$CONJ^i \equiv$

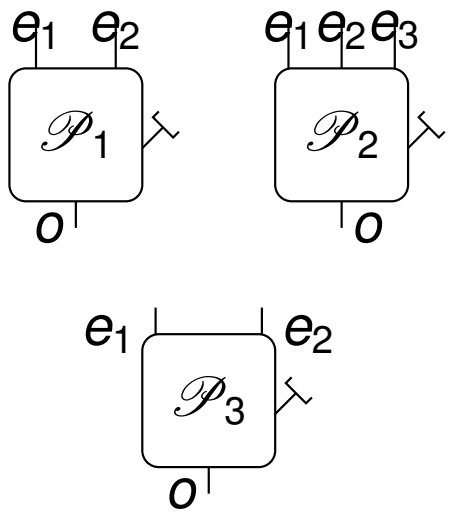


Pieces (4/4)

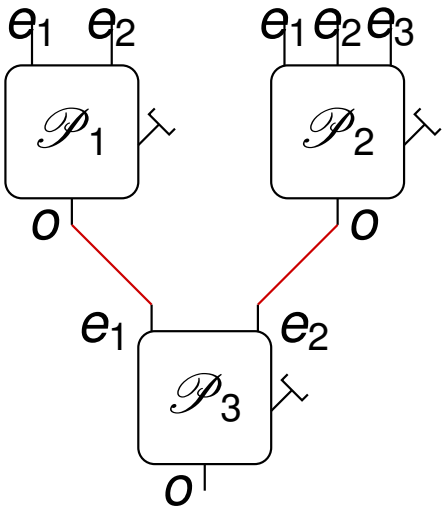
$DUPL^{i} \equiv$



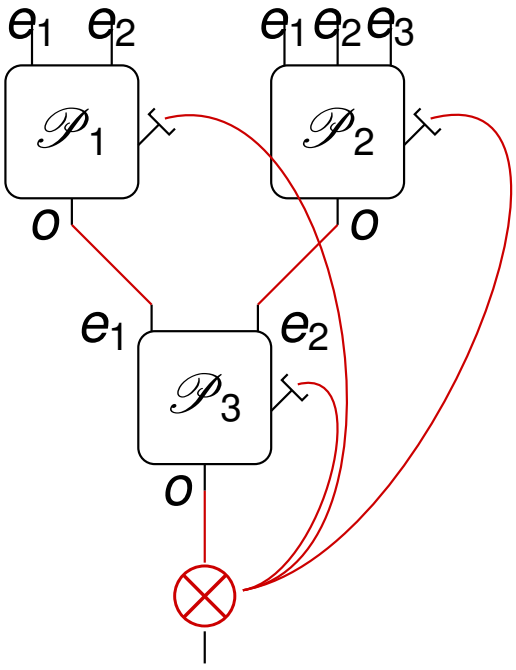
Proof Circuits



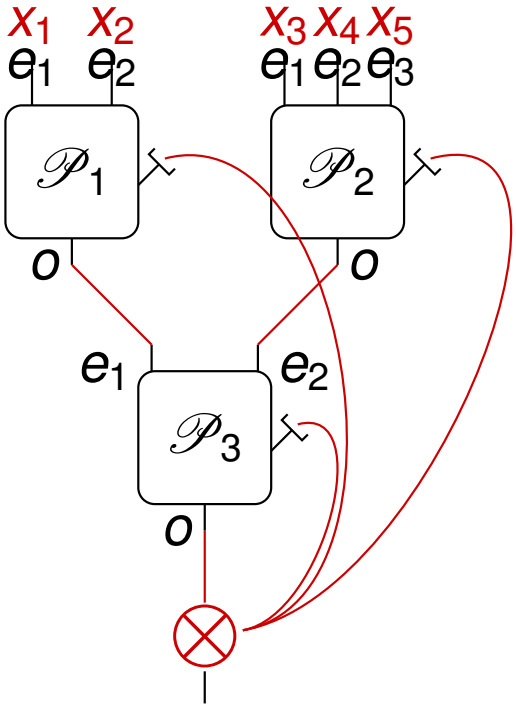
Proof Circuits



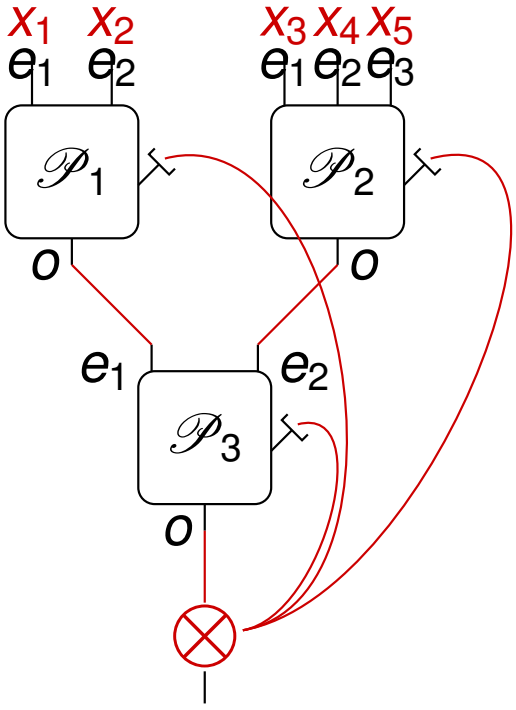
Proof Circuits

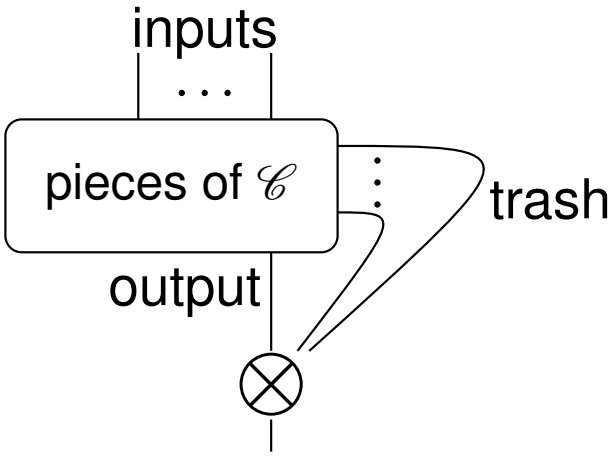


Proof Circuits

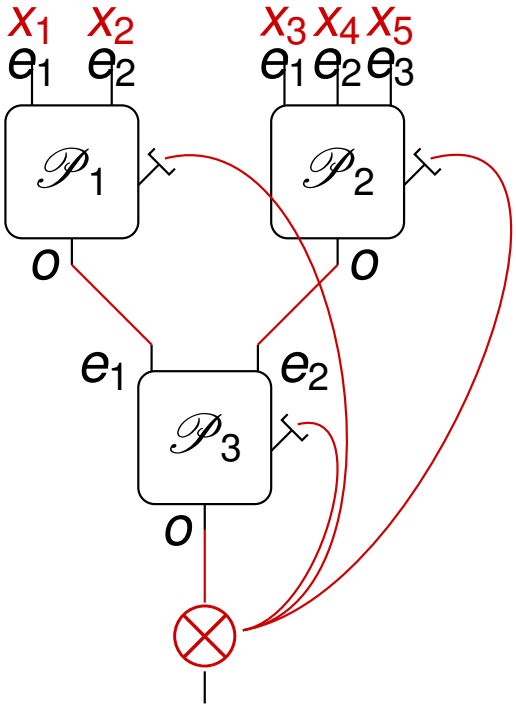


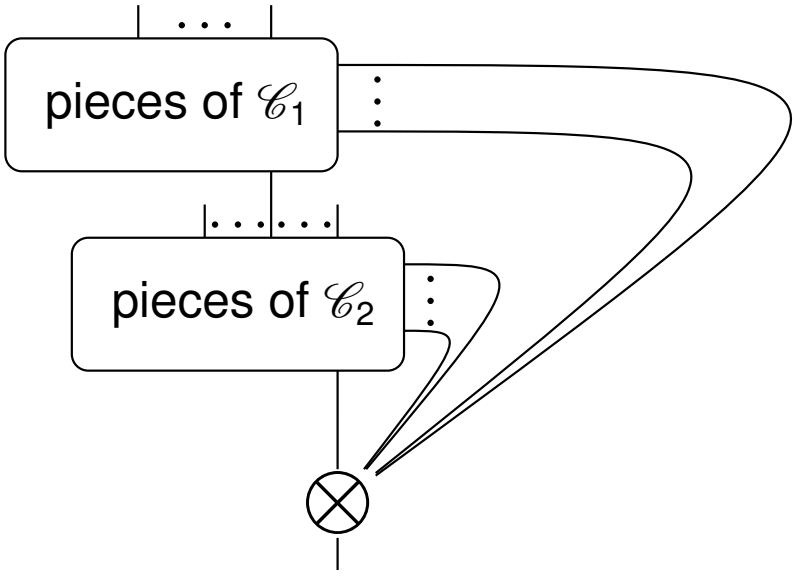
Proof Circuits





Proof Circuits





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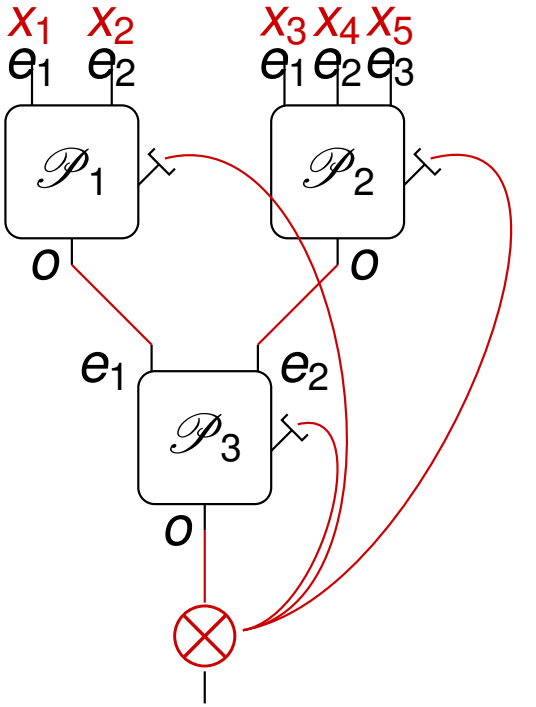

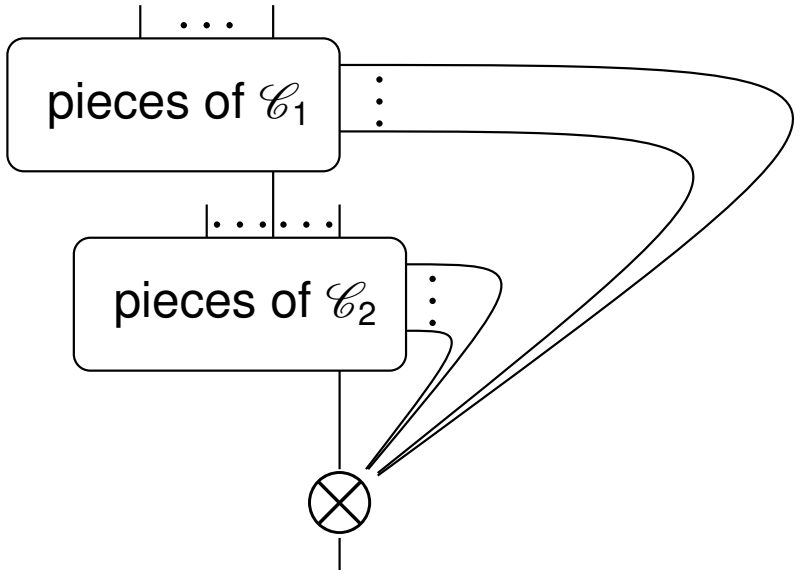

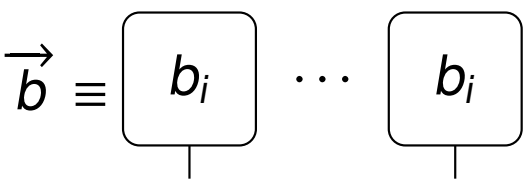

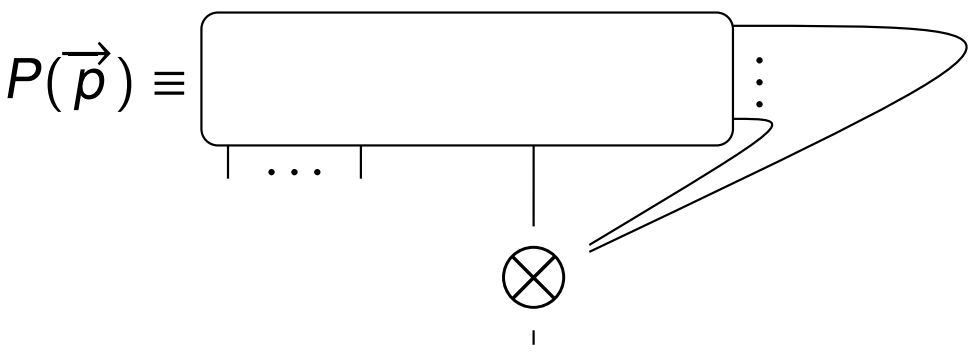

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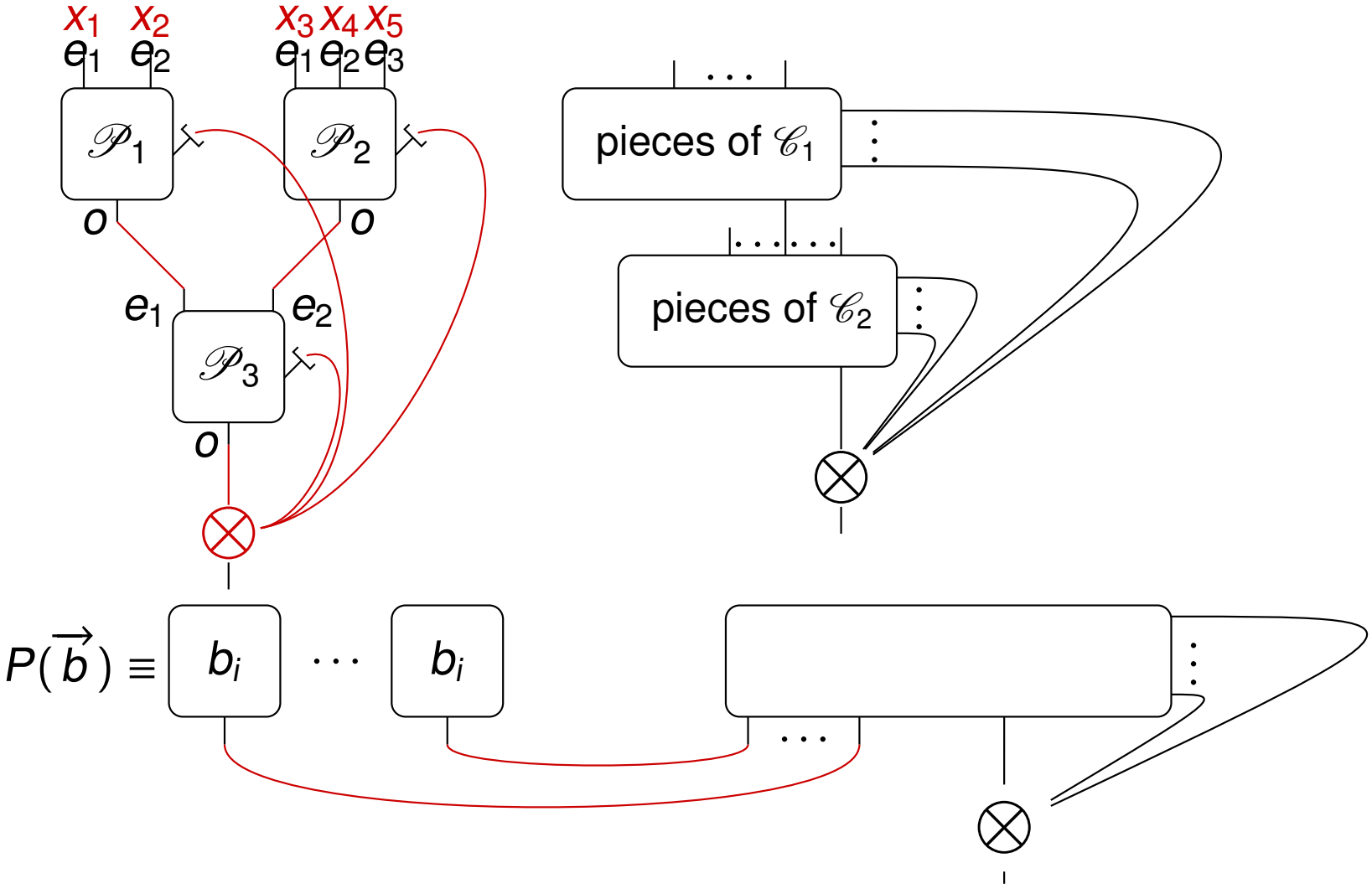

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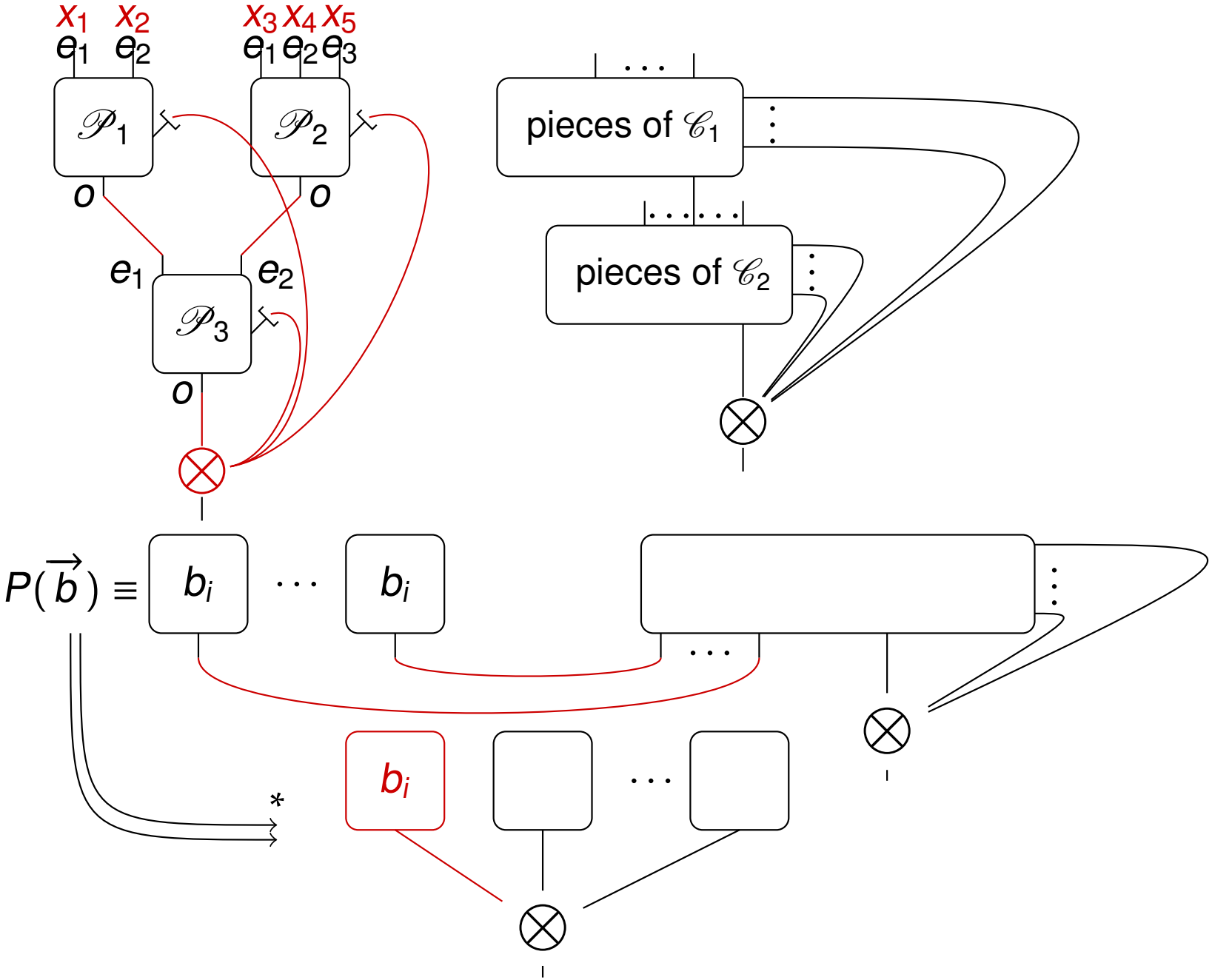


Definition (AC^i)

For $i \in \mathbb{N}$, set of boolean functions computable by a uniform family of Boolean Circuits $C = (C_n)$ where for all C_n

- *its depth is $O(\log^i n)$,*
- *its size is $n^{O(1)}$,*
- *its gate are labeled with functions of $\mathfrak{B}_1 = \{\neg, (\vee^j)_{j\in\mathbb{N}}, (\wedge^j)_{j\in\mathbb{N}}\}$.*

$$\bigcup_{i\in\mathbb{N}} AC^i = AC$$

Formally speaking...

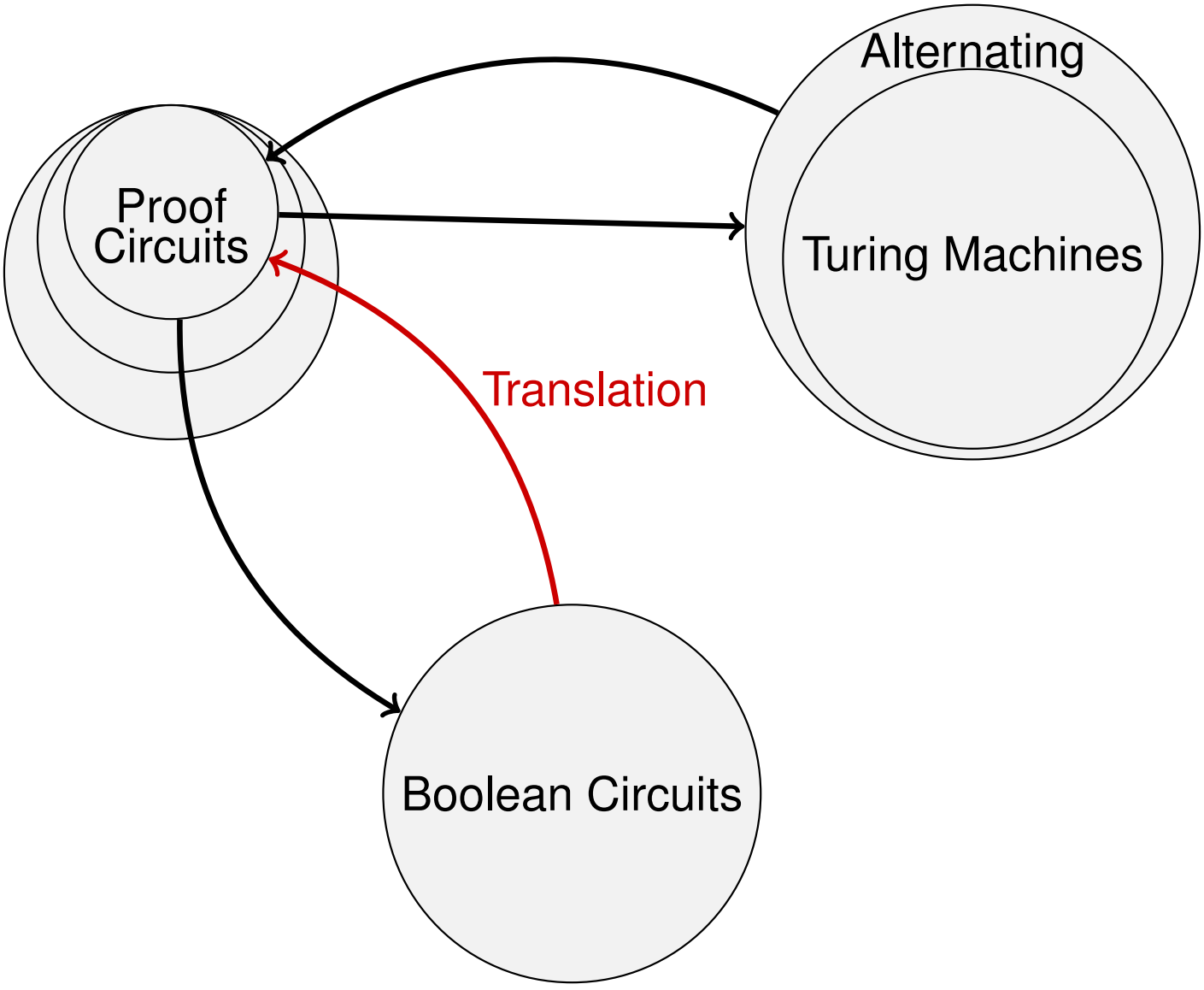
Definition (PCC^i)

For $i \in \mathbb{N}$, set of boolean functions computable by a uniform family of Proof Circuits $P = (P_n)$ where for all P_n

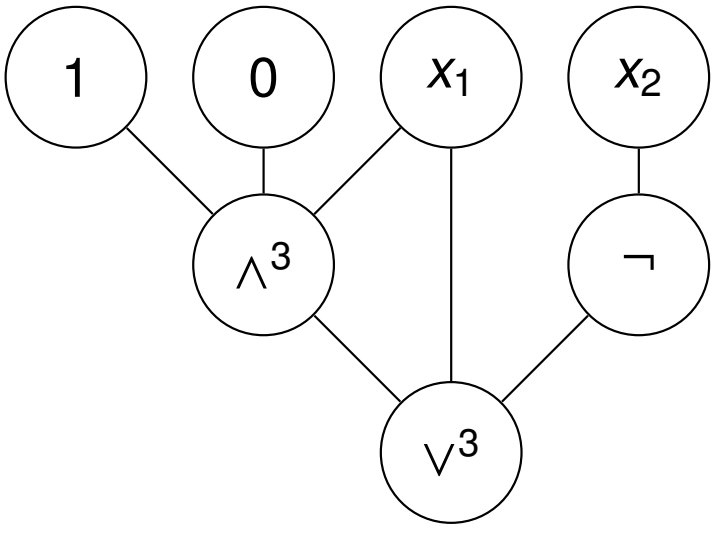
- *its depth is $O(\log^i n)$,*
- *its size is $n^{O(1)}$,*

$$\bigcup_{i \in \mathbb{N}} PCC^i = PCC$$

Map (2~5): From Boolean Circuits to Proof Circuits



Complexity of the translation



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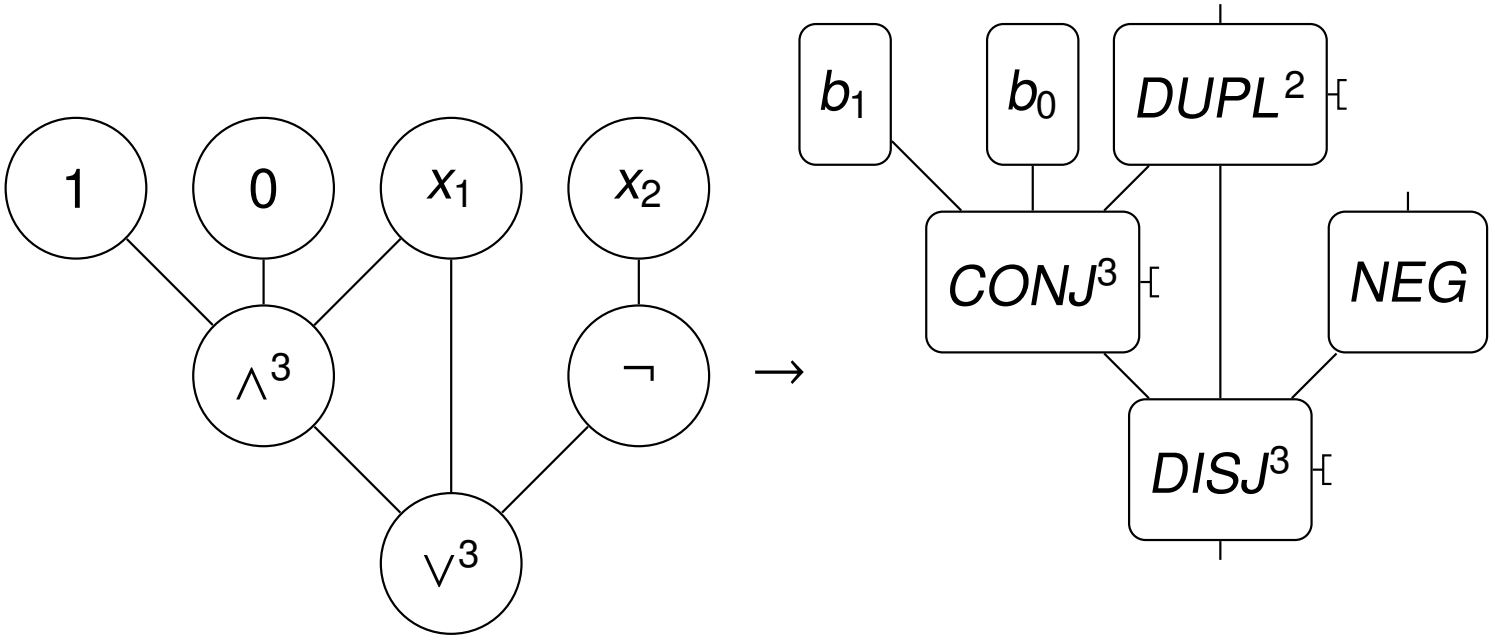

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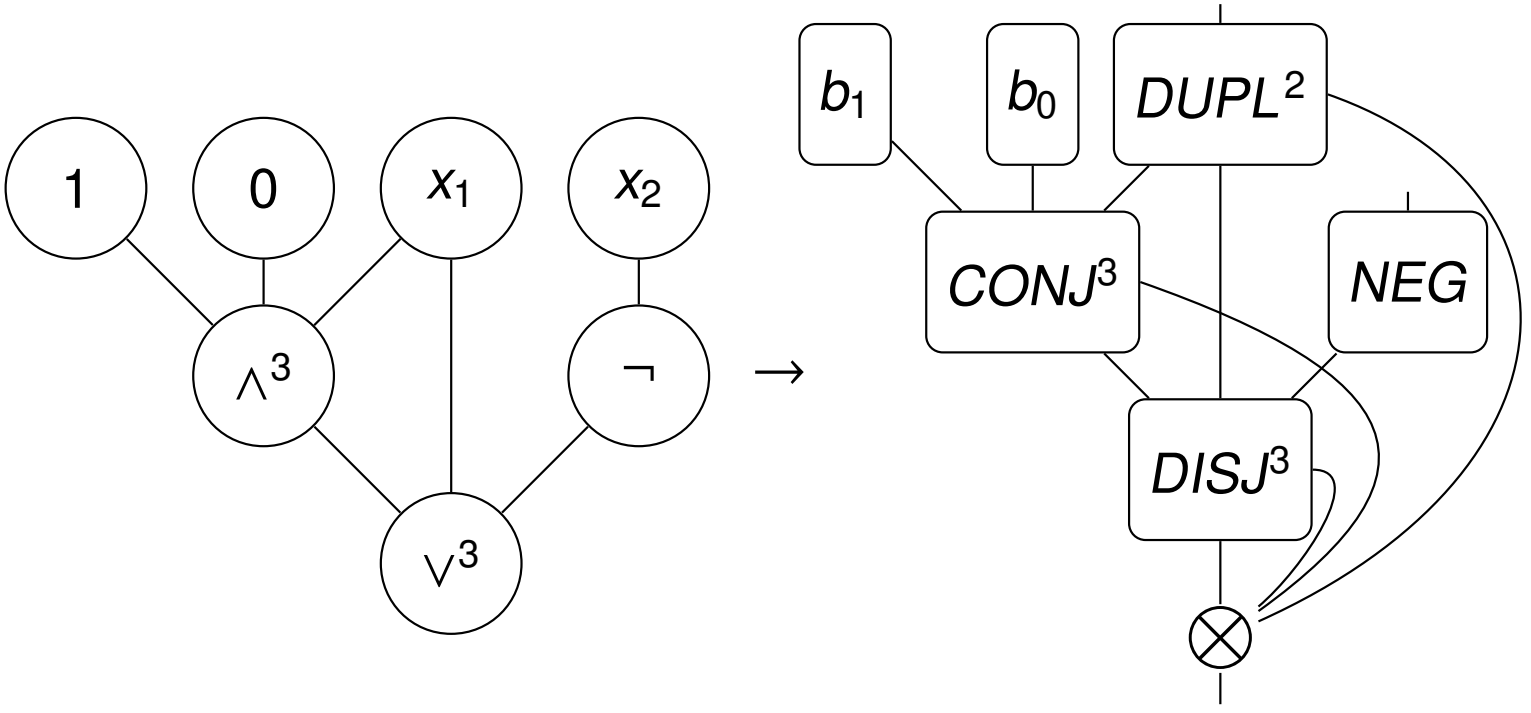

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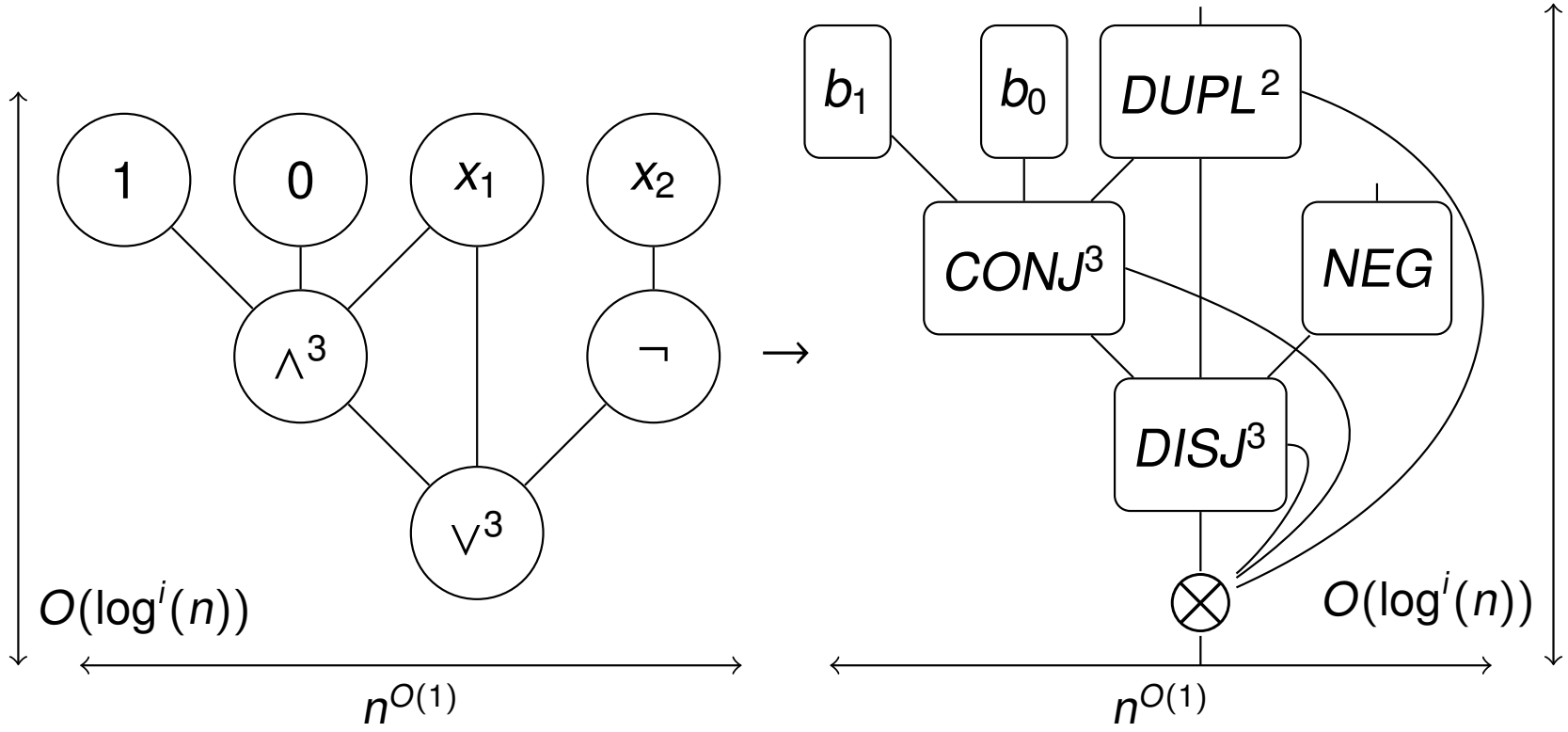


Definition (Translation from AC^i to PCC^i)

Input: *Description of a uniform familly of Boolean Circuits* $C = (C_n)$ *in* AC^i.

Output: *Description of a familly of Proof Circuits* $P = (P_n)$ *in* PCC^i *so that for all* k, *for all* $\overrightarrow{b}$, $P_k(\overrightarrow{b}) \rightarrow_{ev} b_j$ *iff* $C_k(\overrightarrow{b})$ *evaluates to* j.

Theorem

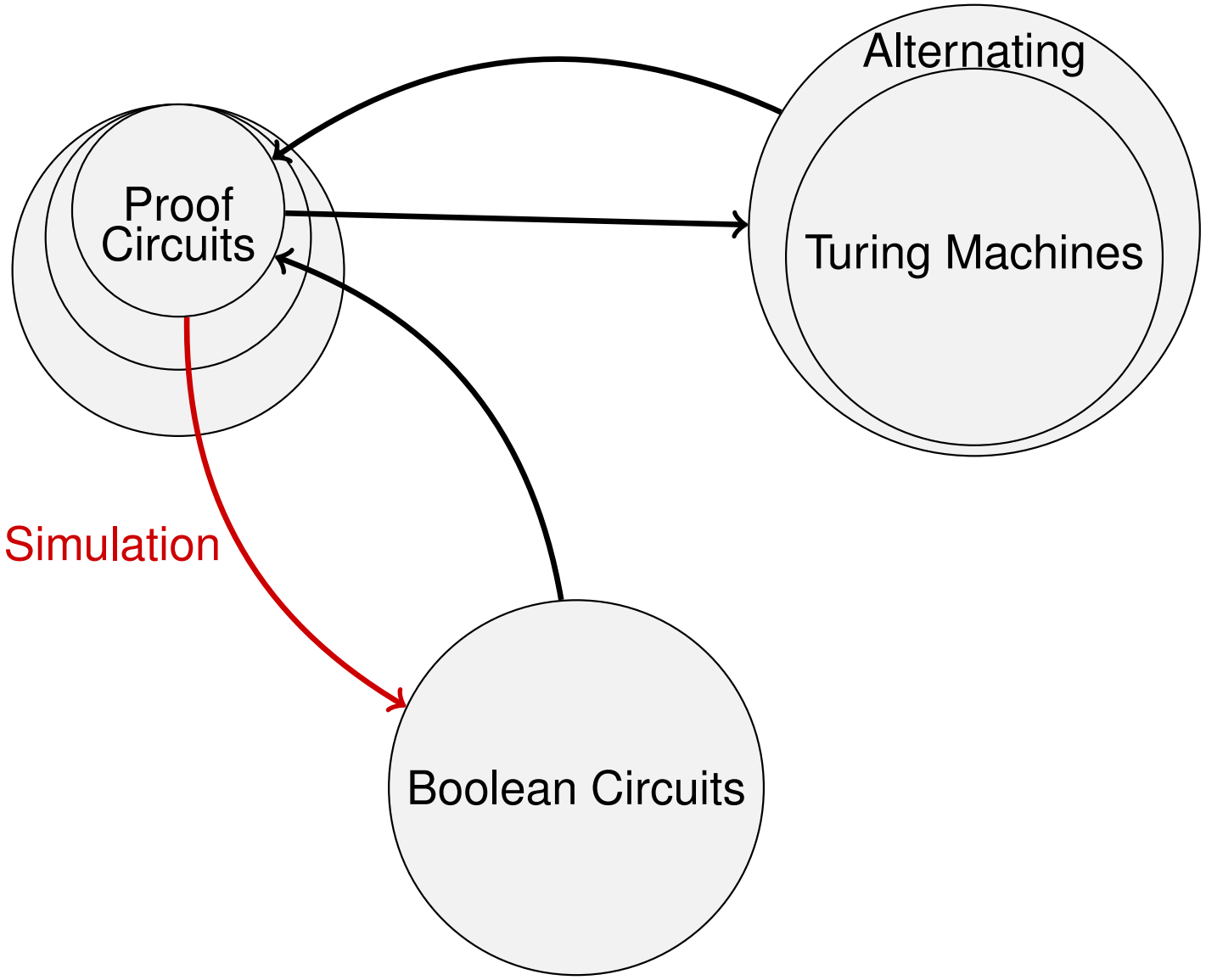
For all $i \in \mathbb{N}$, *Translation from* AC^i *to* PCC^i *belongs to* AC^0.

Theorem (Translation)

For all $i \in \mathbb{N}$,

$$AC^i \subseteq PCC^i$$

Map (3~5): From Proof Circuits to Boolean Circuits



Simulation of $\rightrightarrows$ by Boolean Circuits

Theorem ([Terui, 2004])

Every Proof-Net P normalizes in at most $3\times$ its depth steps of parallel reduction ($\rightrightarrows_t$, $\rightrightarrows_a$, $\rightrightarrows_m$).

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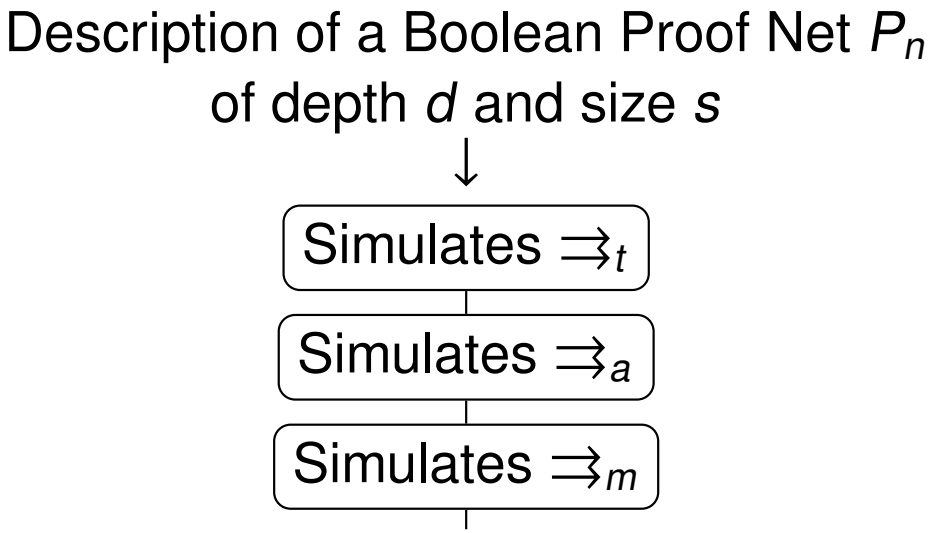

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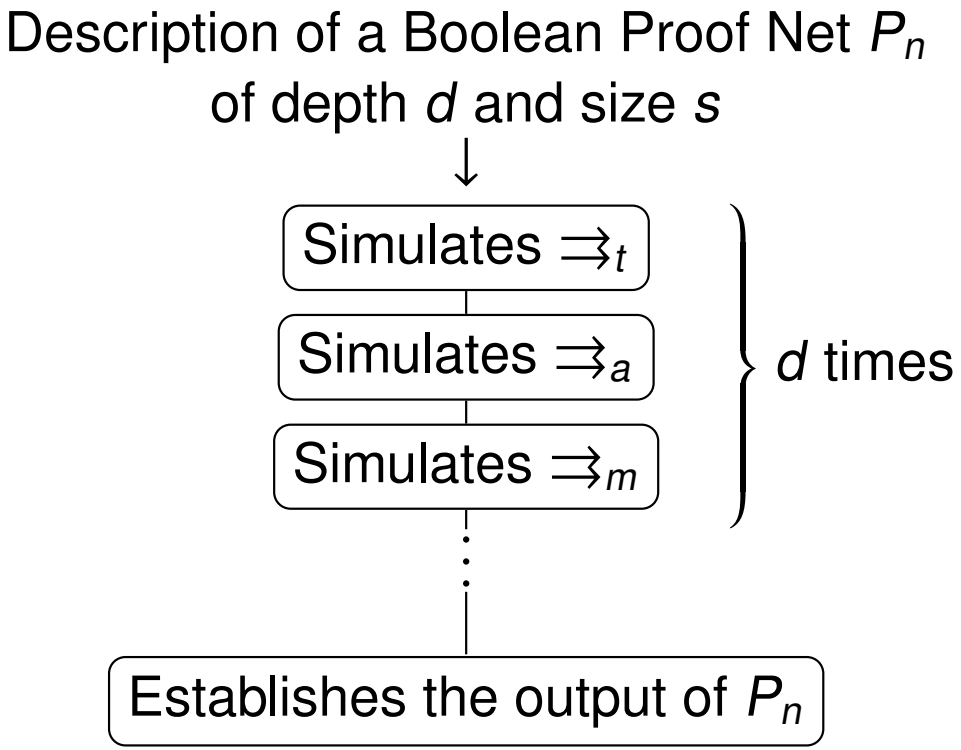

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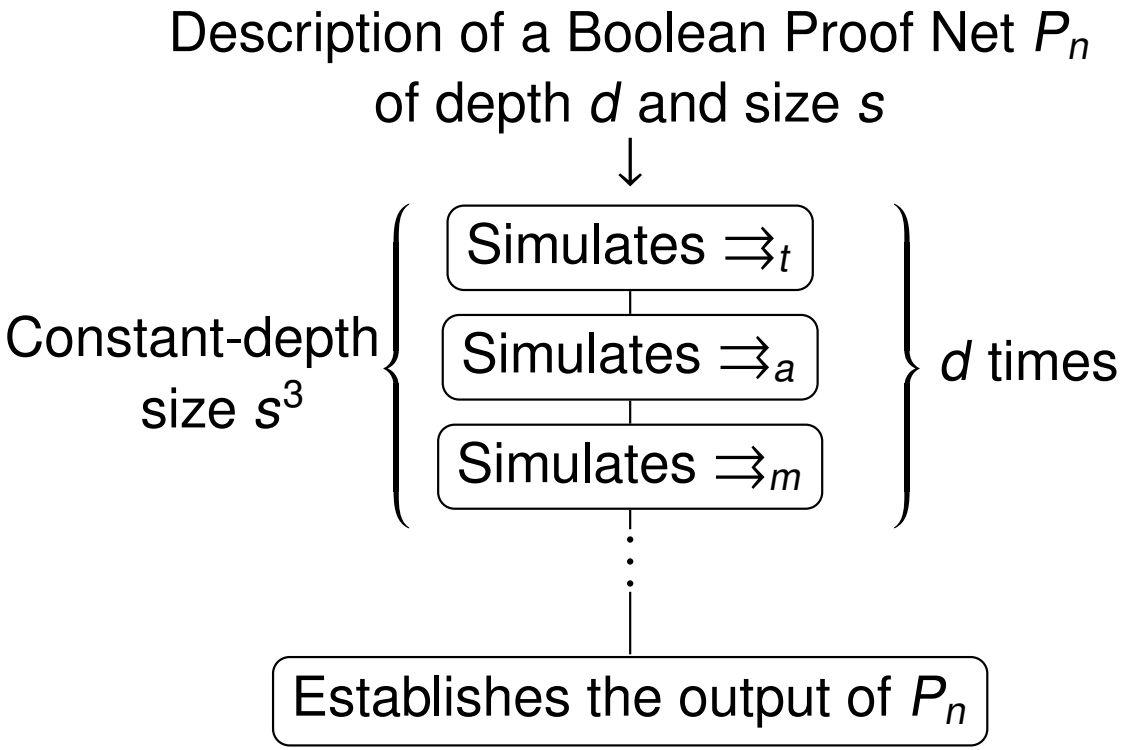
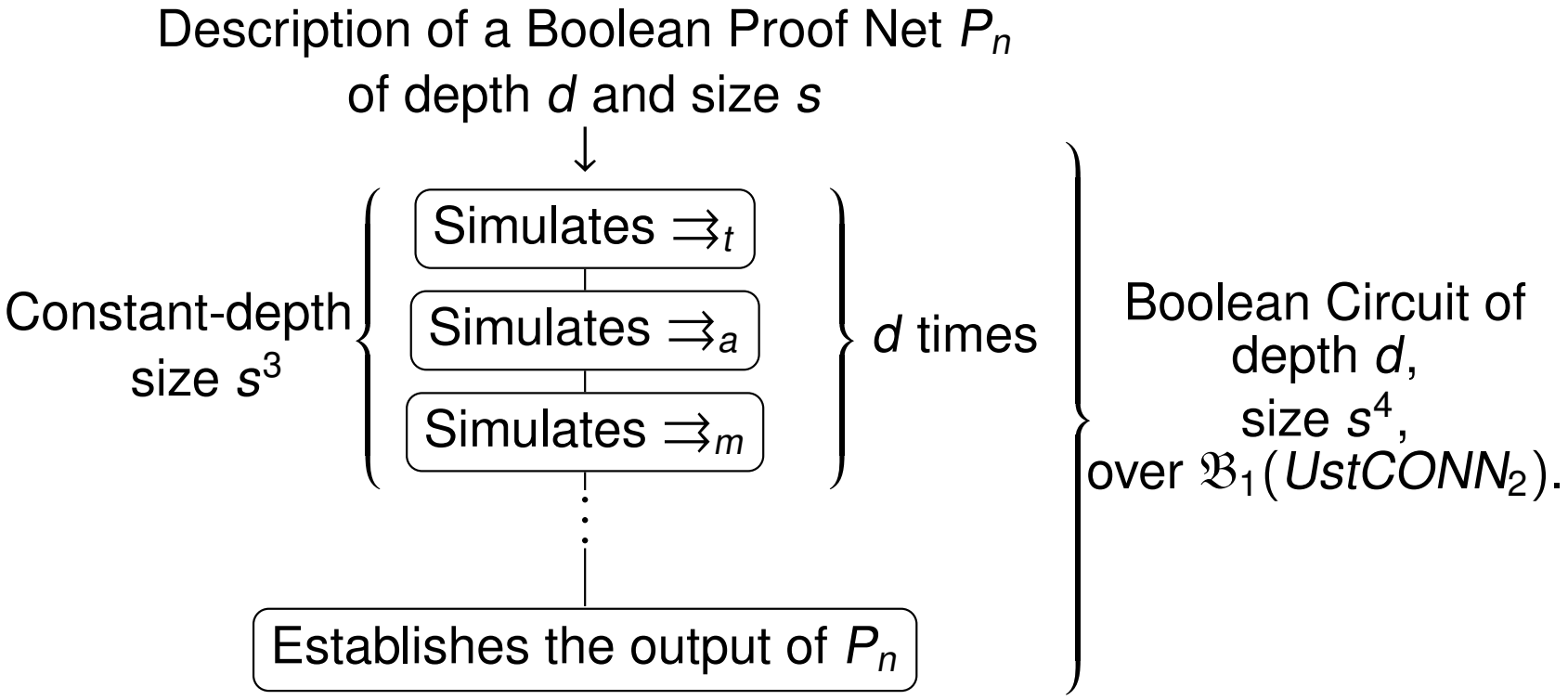


Simulation of $\rightrightarrows$ by Boolean Circuits

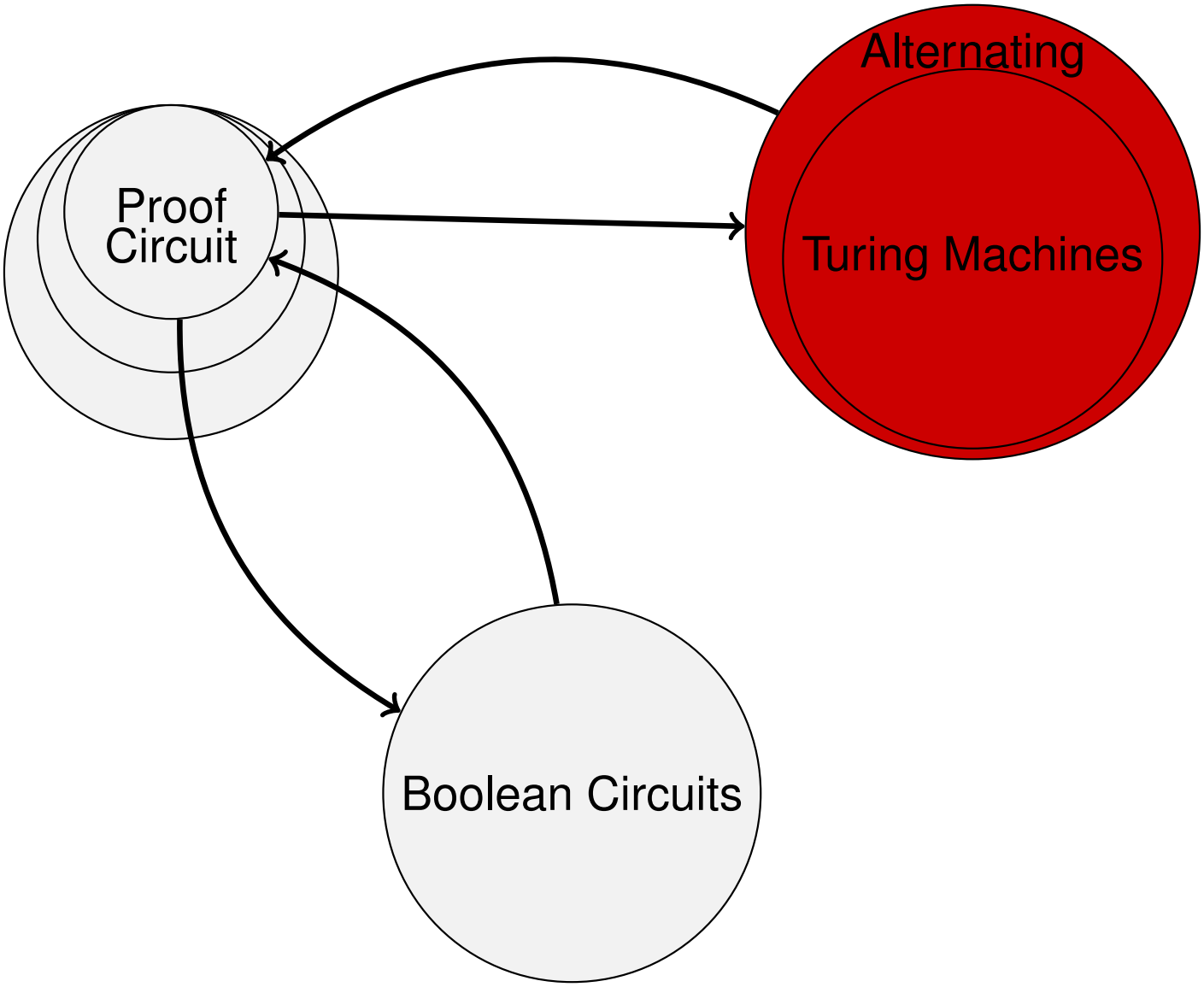
Theorem (Simulation)

For all $i \in \mathbb{N}$,

$$PCC^i \subseteq AC^i(UstCONN_2)$$



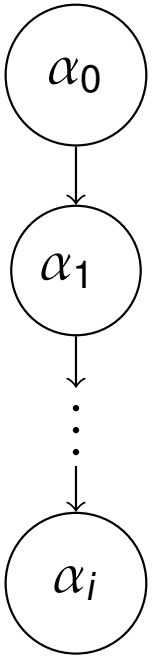
Map (4~5): Alternating Turing Machines



Quick presentation of the ATM

A k-tapes Turing Machine is a tuple $\{K, \Sigma, \delta, s\}$ with

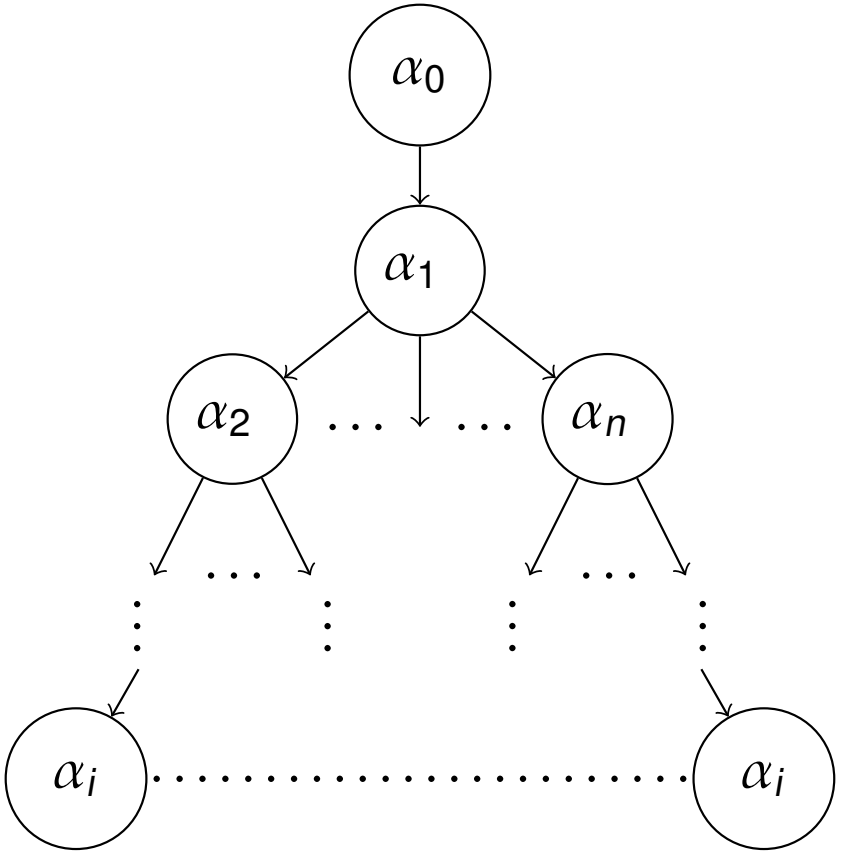
- K a finite set of *states*
- $\Sigma = \{\flat, 0, 1\}$
- δ a function from $K \times \Sigma^K$ to $(K \cup \{a, r\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^K$
- $s \in K$ the initial state



Quick presentation of the ATM

A k-tapes Nondeterminisic Turing Machine is a tuple $\{K, \Sigma, \Delta, s\}$ with

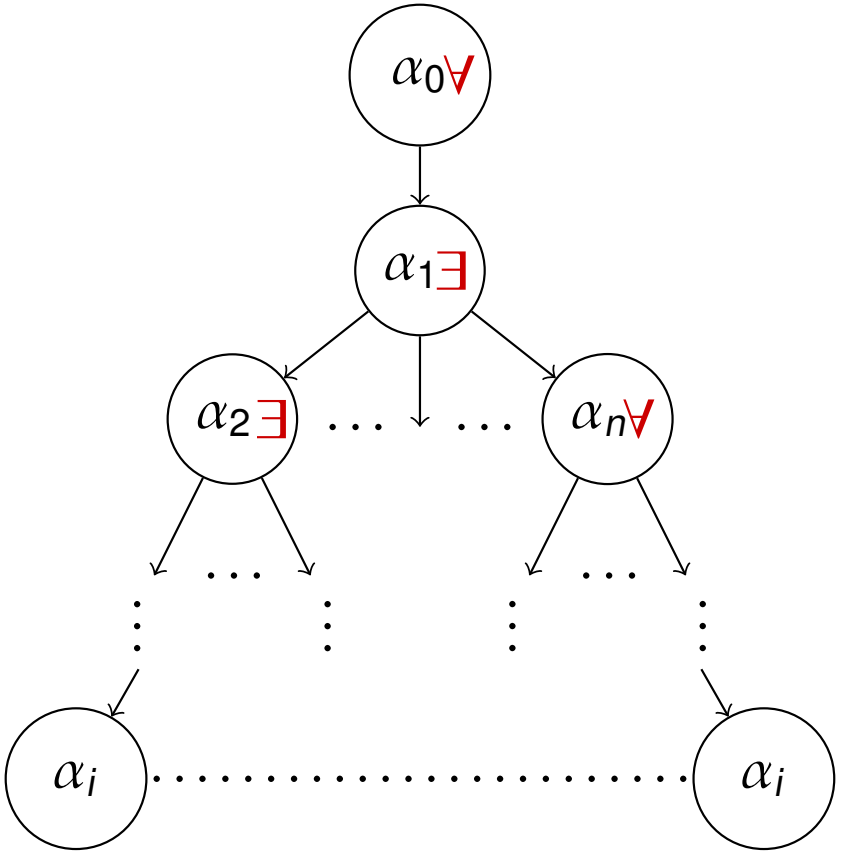
- K a finite set of *states*
- $\Sigma = \{\flat, 0, 1\}$
- Δ a relation $\subset (K \times \Sigma) \times [(K \cup \{a, r\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}]$
- $s \in K$ the initial state



Quick presentation of the ATM

A k-tapes Alternating Turing Machine is a tuple $\{K, \Sigma, \Delta, s\}$ with

- K a finite set of *states*, partitionned into 2 sets, $K_\forall$ and $K_\exists$
- $\Sigma = \{\flat, 0, 1\}$
- Δ a relation $\subset (K \times \Sigma) \times [(K \cup \{a, r\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}]$
- $s \in K$ the initial state



Quick presentation of the ATM

A k-tapes Alternating Turing Machine is a tuple $\{K, \Sigma, \Delta, s\}$ with

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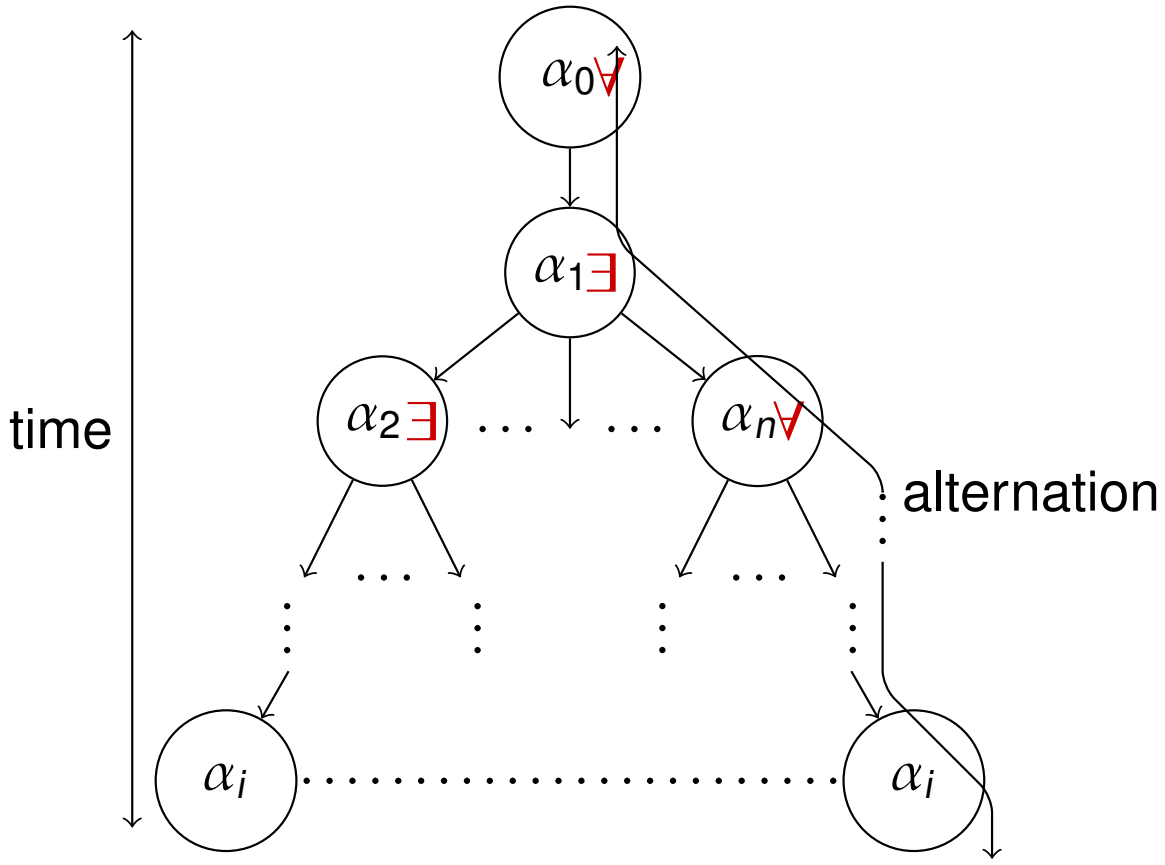
- $s \in K$ the initial state



Quick presentation of the ATM

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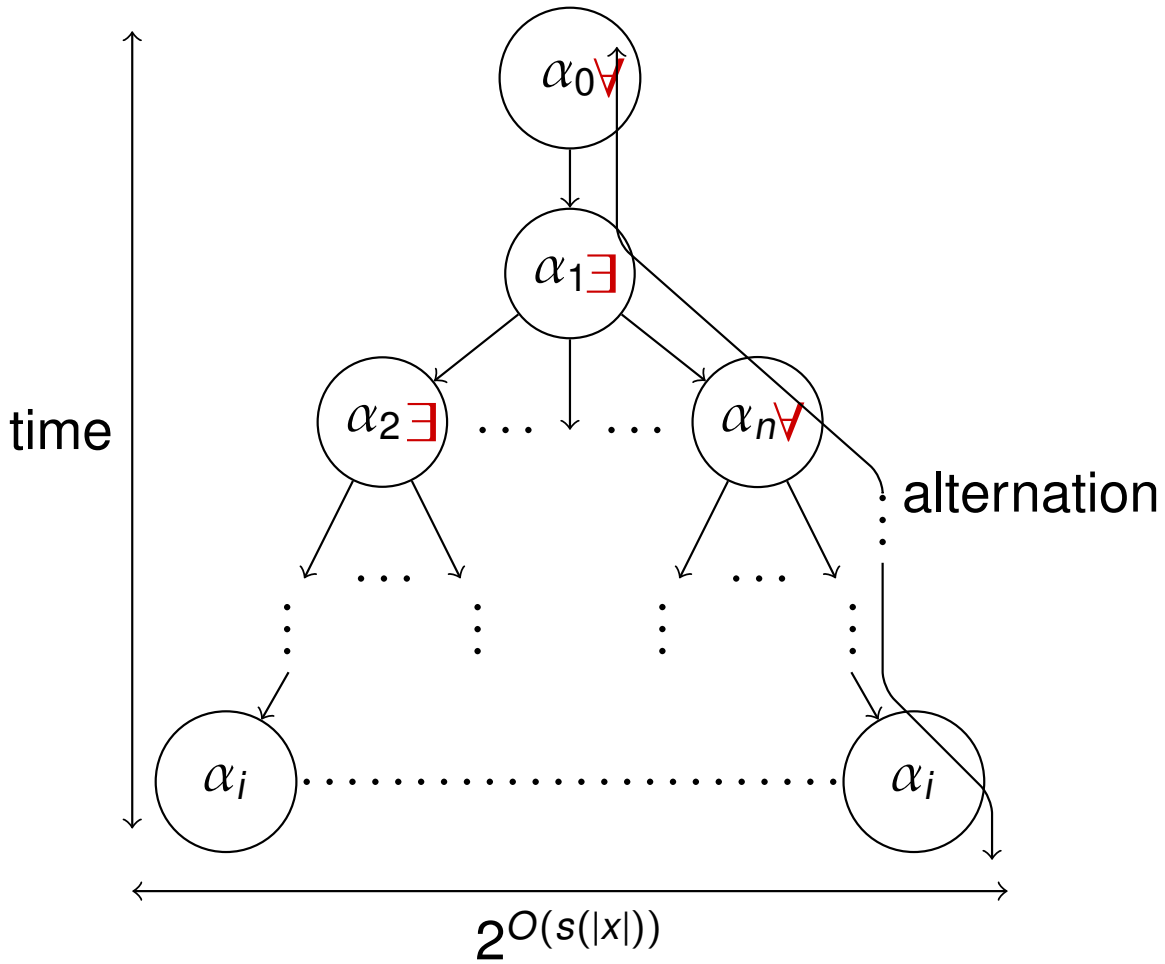
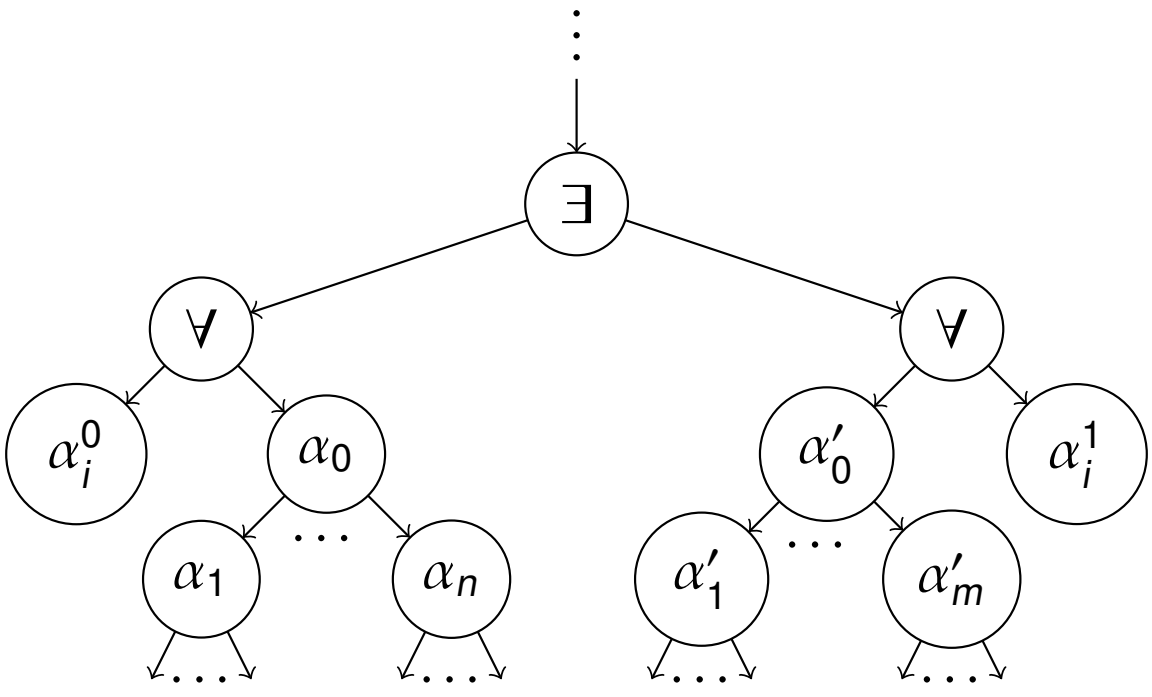
- $s \in K$ the initial state



Transformations on graphs of ATM

The aim is to obtain a finite tree that represents the behaviour of an ATM at given size of input.

- Input Normal Form



α_i^j is an accepting state iff $x_i = j$.

Transformations on graphs of ATM

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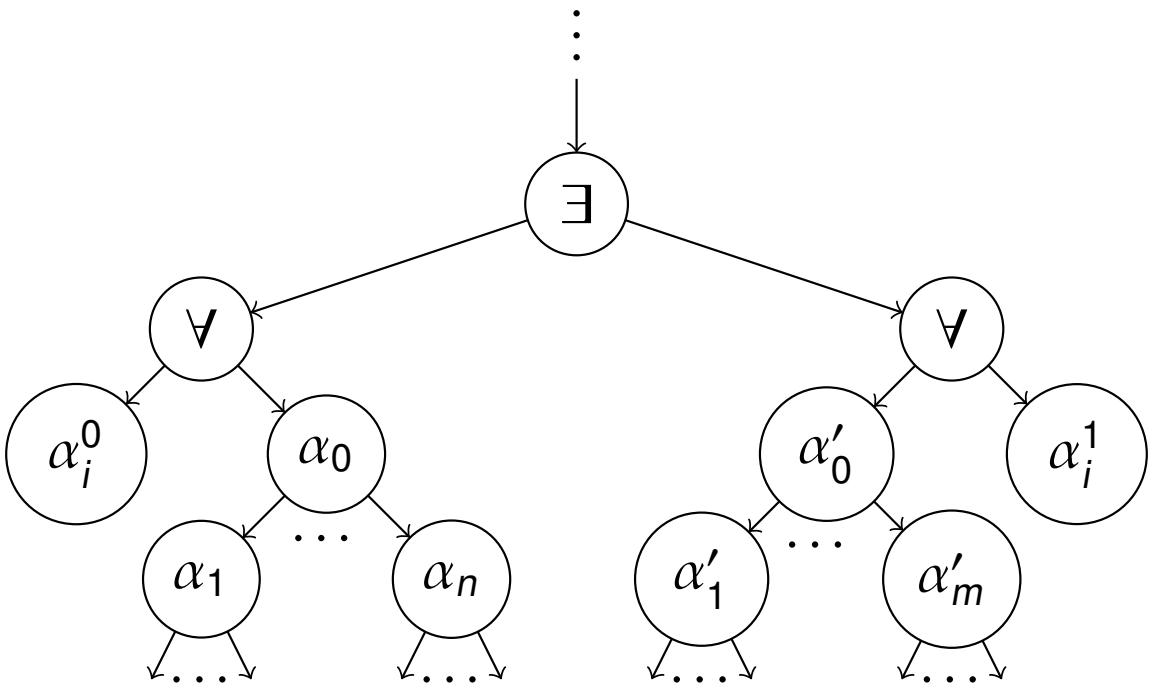
- Input Normal Form
- Clock: Remove cycles and infinite branchs



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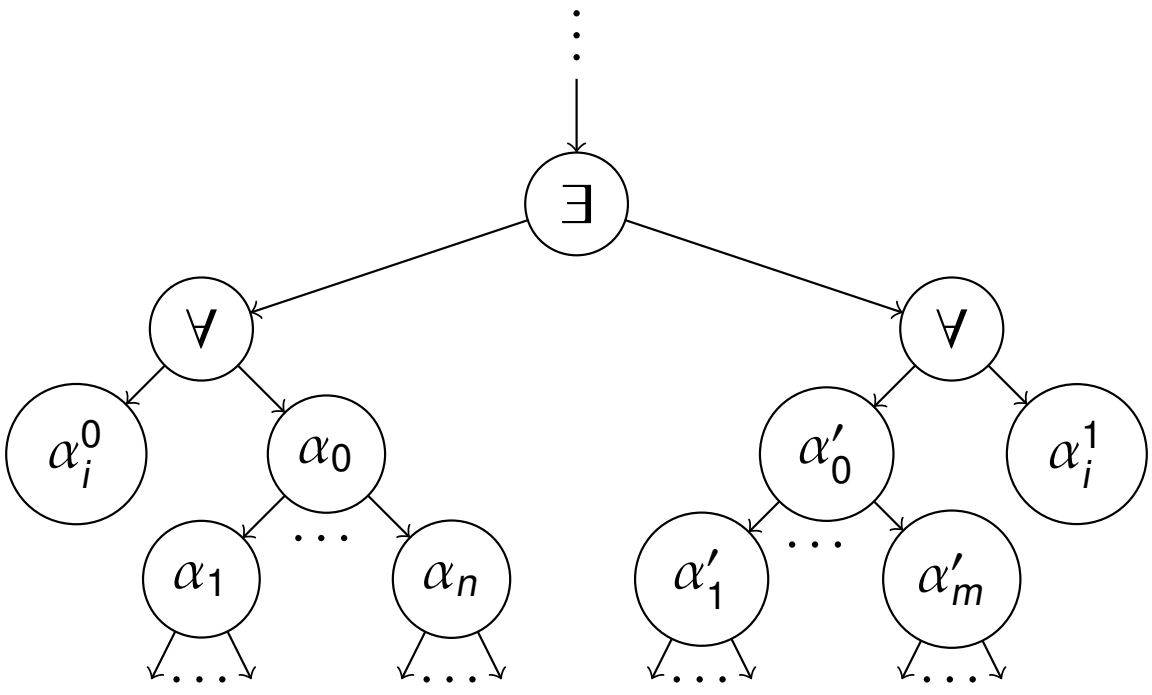
Transformations on graphs of ATM

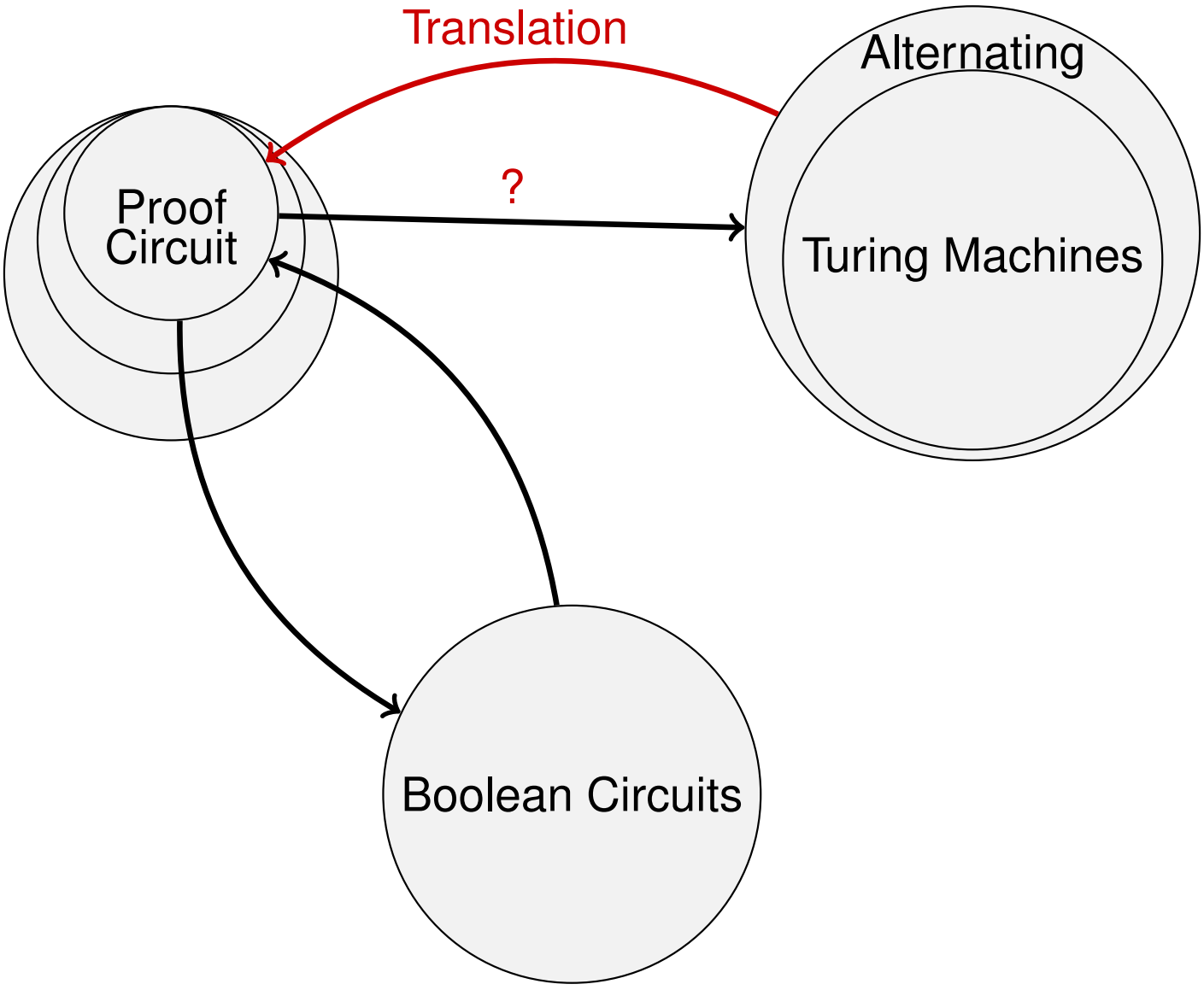
The aim is to obtain a finite tree that represents the behaviour of an ATM at given size of input.

- Input Normal Form
- Clock: Remove cycles and infinite branchs
- Every configuration has at most one predecessor

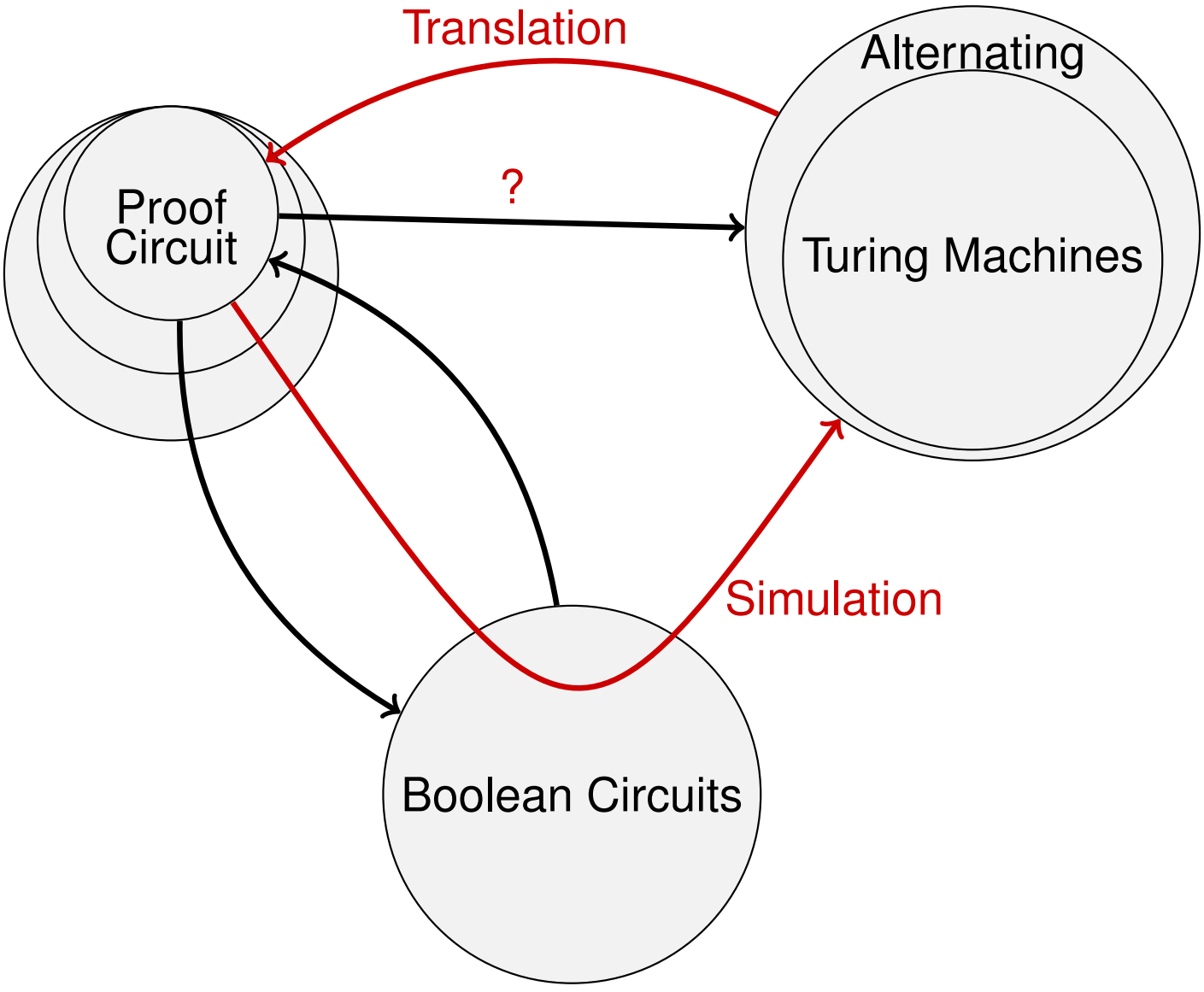


α_i^j is an accepting state iff $x_i = j$.

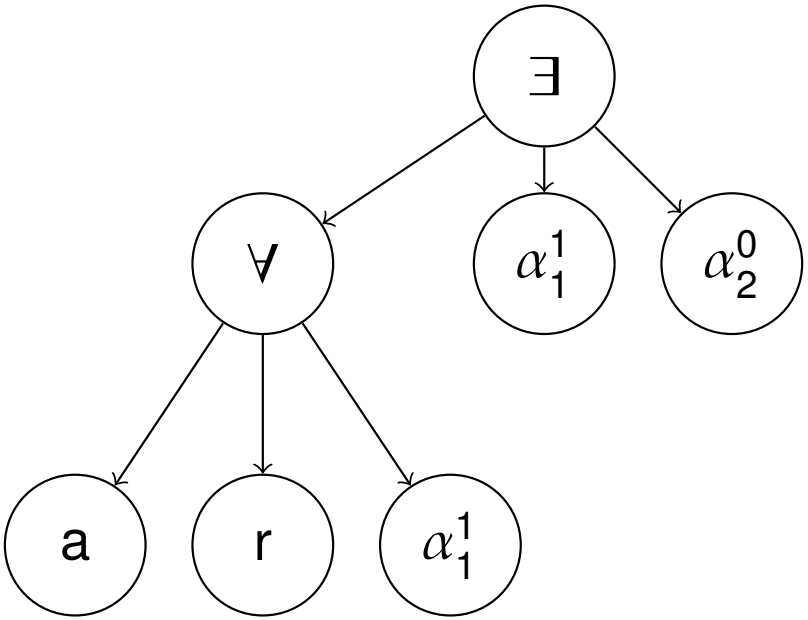
Map (5~5): A.T.M. and Proof Circuits



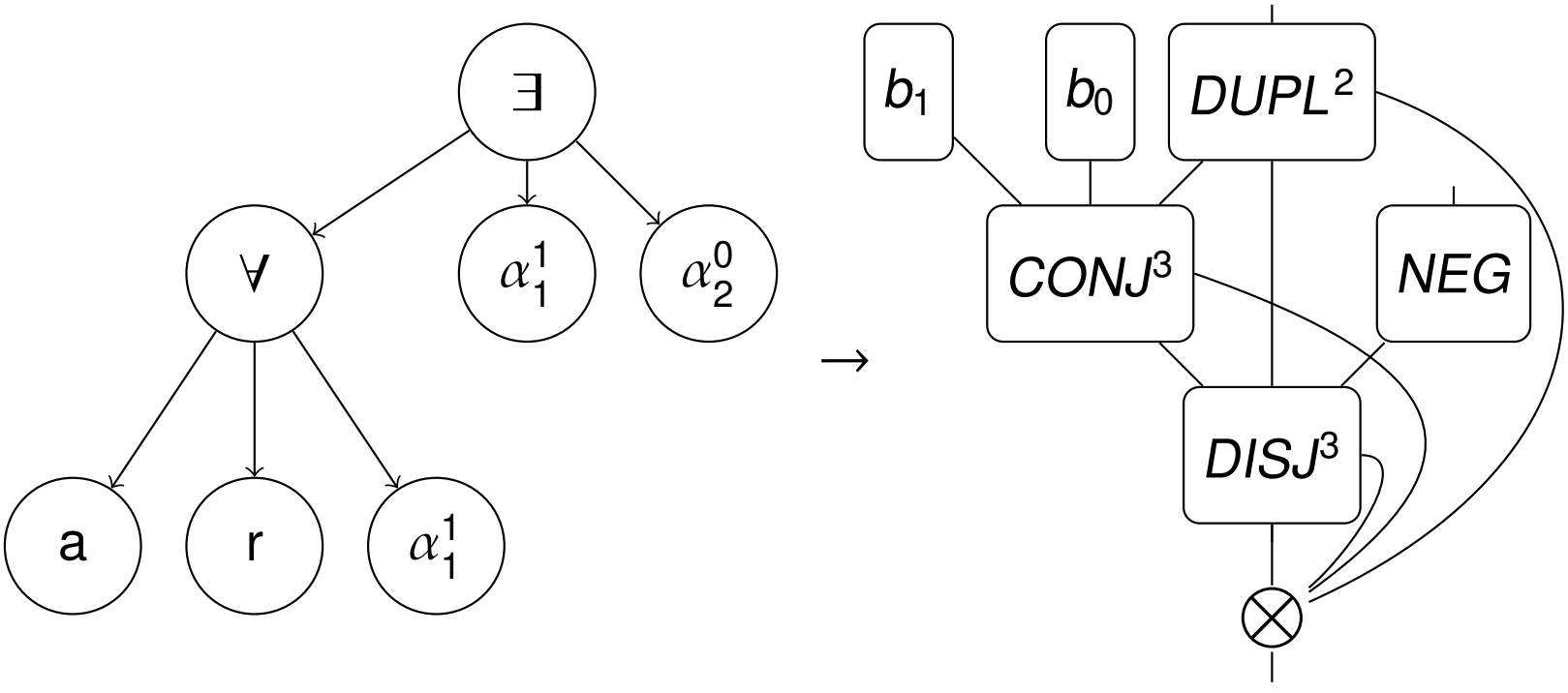
Map (5~5): A.T.M. and Proof Circuits



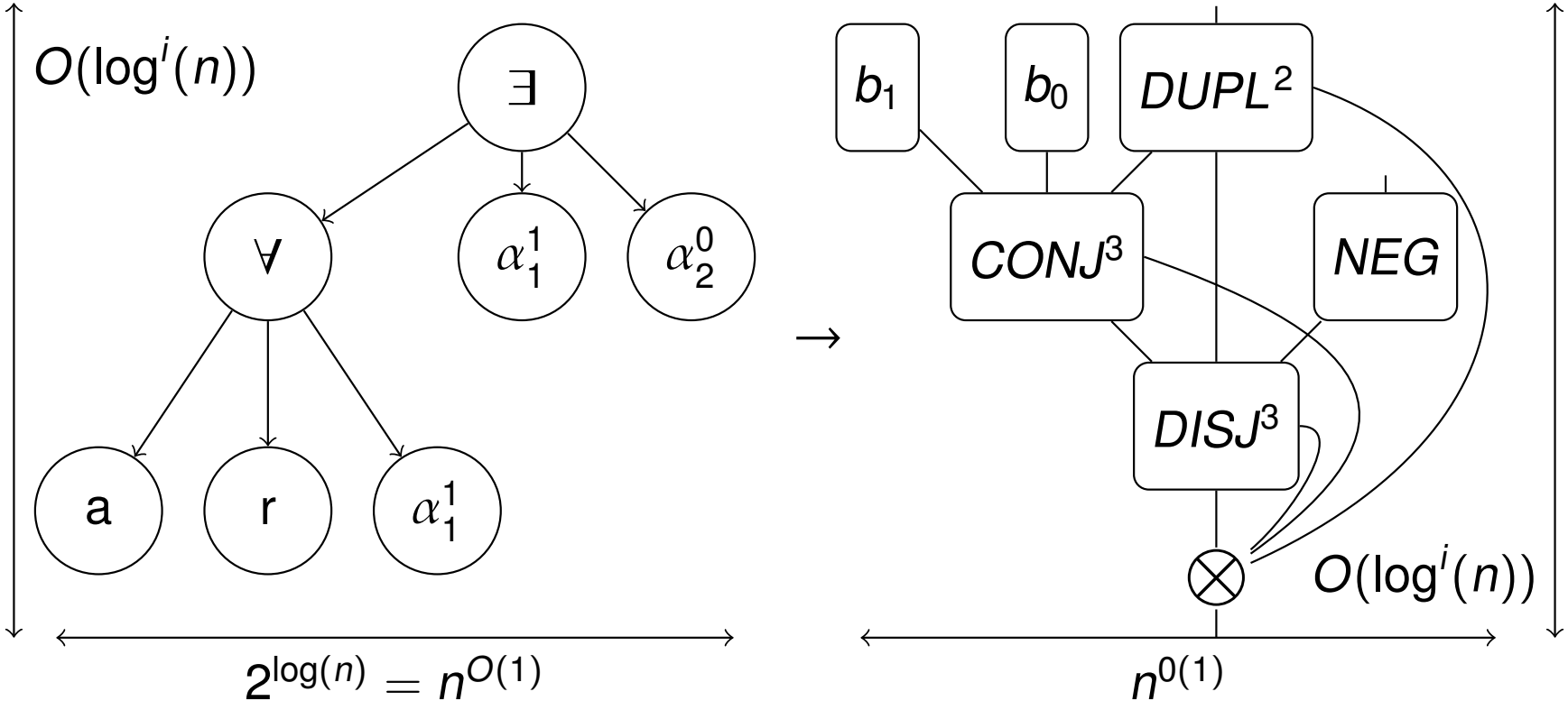
Translation from A.T.M to P.C.



Translation from A.T.M to P.C.



Translation from A.T.M to P.C.



Definition (STA)

$STA(s, t, a)$ for s, t and a proper complexity functions is the set of boolean functions computable by an *ATM* using space s, time t and alternation a.

Theorem (Translation)

For $i \geqslant 2$,

$$STA(\log(n), \log^i(n), *) \subseteq PCC^i$$

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$STA(s, t, a)$ for s, t and a proper complexity functions is the set of boolean functions computable by an *ATM* using space s, time t and alternation a.

Theorem (Translation)

For $i \geqslant 2$,

$$STA(\log(n), \log^i(n), *) \subseteq PCC^i$$

Theorem (Simulation through Boolean Circuits, inspired by [Vollmer, 1999])

For $i \geqslant 2$,

$$PCC^i \subseteq STA(\log(n), \log^{i+2}, *)$$

That's all folks!


Aubert, C. (2011).
Sublogarithmic uniform boolean proof nets.
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Vollmer, H. (1999).
Introduction to Circuit Complexity: A Uniform Approach.
Springer Verlag.

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