Using Stochastic Comparison for Efficient Model Checking of Uncertain Markov Chains [1]

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Motivation

From probabilistic discrete-event systems to Markov chains (MCs)

- Probabilistic systems are not necessarily memoryless (timeouts, packet arrivals, etc.).
- However during the modeling and analysis process, one often encounters Markov chains (e.g. the embedded Markov chain of a semi-Markovian process).

Why Interval Markov Chains (IMCs)?

- Estimation of the transition rates through statistical experiences leading to confidence intervals.
- Abstraction of events during the modeling step or abstraction of states during the analysis step.

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Analysis of IMC

First works

- Introduction of the formalism and study of conformance relations between models.
  (Jonsson, Larsen LCS'91)
- Methods for computing the parameters of an IMC.
  (Kozine, Utkin Reliable Computing 2002)

Probabilistic model-checking

- Analysis of the model checking of PCTL over IMCs: in PSPACE (via the existential theory of reals), NP-hard and coNP-hard.
  (Sen, Wiswanathan, Agha TACAS'06)
- Generalization for a new logic ω-PCTL: still in PSPACE.
  (Chatterjee, Sen, Henzinger FOSSACS'08).

Handling efficiently model checking for IMC

Drawbacks: complexity and expressivity considerations

- Algorithms in PSPACE are impractical for large IMCs.
- Some useful properties cannot be expressed even with ω-PCTL.

Goal: semi-decision procedures based on stochastic comparison

- Generally different magnitude orders between the requirement and implementation probabilities.
  Thus the *don't know* case should seldom occur.
- The problem is reduced to the model checking of MCs.
  This should lead to a significant improvement w.r.t. time complexity.

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Outline

1. IMC model
2. PCTL
3. Efficient Model Checking PCTL for IMCs
4. Conclusion and perspectives

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1. IMC model

PCTL

Efficient Model Checking PCTL for IMCs

Conclusion and perspectives

Interval Markov Chain

Syntax

An IMC $\mathcal{M}(\mathbf{P}^-, \mathbf{P}^+) = (\mathcal{S}, \mathbf{P}^-, \mathbf{P}^+, L)$ is defined by:

- $\mathcal{S}$, the finite set of states which are labelled by atomic properties through the mapping L;
- $\mathbf{P}^-$ (resp. $\mathbf{P}^+$ with $\mathbf{P}^+ \geq \mathbf{P}^-$), a sub-stochastic (resp. super-stochastic) matrix:

$$\forall s \in \mathcal{S} \;\; \sum_{t \in \mathcal{S}} \mathbf{P}^-[s,t] \leq 1 \leq \sum_{t \in \mathcal{S}} \mathbf{P}^+[s,t]$$

Semantic

A DTMC with transition probability matrix $\mathbf{P}$ over $\mathcal{S}$ is said to belong to $\mathcal{M}(\mathbf{P}^-, \mathbf{P}^+)$ (denoted $\mathbf{P} \in \mathcal{M}(\mathbf{P}^-, \mathbf{P}^+)$), if:

$$\forall s, t \in \mathcal{S} \quad \mathbf{P}^-[s,t] \leq \mathbf{P}[s,t] \leq \mathbf{P}^+[s,t]$$

W.l.o.g. we assume that:

$$\mathbf{P}^-[s,t] \geq 1 - \sum_{t' \neq t} \mathbf{P}^+[s,t'] \wedge \mathbf{P}^+[s,t] \leq 1 - \sum_{t' \neq t} \mathbf{P}^-[s,t']$$

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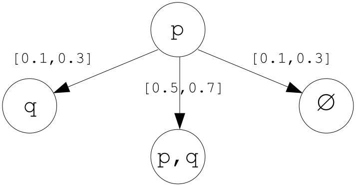
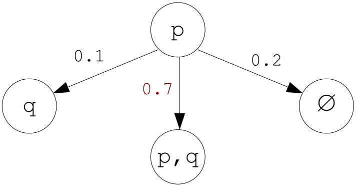
An IMC with two associated DTMCs



Maximizing a probability transition



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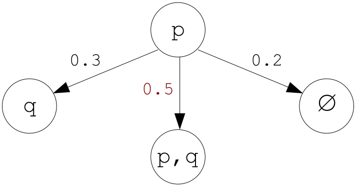

Optimal values for cumulative transition probabilities

Individual transition probabilities

Bounds can always be reached. For every $s, t \in \mathcal{S}$, there is a $\mathbf{P} \in \mathcal{M}(\mathbf{P}^-, \mathbf{P}^+)$ with $\mathbf{P}[s,t] = \mathbf{P}^+[s,t]$ and a $\mathbf{P} \in \mathcal{M}(\mathbf{P}^-, \mathbf{P}^+)$ with $\mathbf{P}[s,t] = \mathbf{P}^-[s,t]$.

Let $s \in \mathcal{S}$ and $\mathcal{S}' = \{s_1, \ldots, s_m\} \subset \mathcal{S} = \{s_1, \ldots, s_n\}$

How to maximize $\sum_{t \in \mathcal{S}'} \mathbf{P}[s,t]$ for possible $\mathbf{P}$ in $\mathcal{M}(\mathbf{P}^-, \mathbf{P}^+)$?

- Maximize one by one the probability transition taking into account the constraints updated by the previous choices.
- More formally, let $sum = \sum_{j<i} \mathbf{P}[s, s_j]$. Then: $\mathbf{P}[s, s_i] = \min(\mathbf{P}^+[s, s_i], 1 - sum - \sum_{j>i} \mathbf{P}^-[s, s_j]);$

Observations

- There is a similar algorithm for minimization.
- Different subrows $\mathbf{P}[s, -]$ are possible depending on the ordering of $\mathcal{S}'$.

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IMC for sub-stochastic matrices

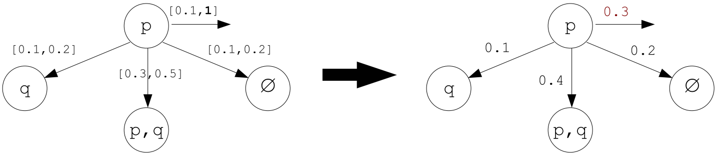
When model checking MCs, one produces MCs with an absorbing state or equivalently sub-stochastic matrices. So:

An IMC $\mathcal{M}(\mathbf{P}^-, \mathbf{P}^+, \mathbf{out})$ for sub-stochastic matrices is enlarged with a vector $\mathbf{out}$ over states such that $\mathbf{P}^-, \mathbf{P}^+, \mathbf{out}$ fulfill for all $s, t \in \mathcal{S}$:

- $0 \leq \mathbf{P}^-[s,t] \leq \mathbf{P}^+[s,t] \wedge \sum_{t' \in \mathcal{S}} \mathbf{P}^-[s,t'] + \mathbf{out}[s] \leq 1$
- $\mathbf{P}^+[s,t] \leq 1 - \sum_{t' \neq t} \mathbf{P}^-[s,t'] - \mathbf{out}[s]$

A sub-stochastic matrix $\mathbf{P}$ over $\mathcal{S}$ belongs to $\mathcal{M}(\mathbf{P}^-, \mathbf{P}^+, \mathbf{out})$ if:

$$\forall s, t \in \mathcal{S} \quad \mathbf{P}^-[s,t] \leq \mathbf{P}[s,t] \leq \mathbf{P}^+[s,t] \wedge \sum_{t' \neq t} \mathbf{P}[s,t'] \leq 1 - \mathbf{out}[s]$$



Stochastic bounds

The survival toolkit: definitions

- Let X, Y be two defective distributions over $\mathcal{S} = \{s_1, \ldots, s_n\}$ defined by $p_X(i) = prob(X = s_i)$ and $p_Y(i) = prob(Y = s_i)$. Then:
$$X \leq_{st} Y \text{ if } \forall i \ \ \sum_{k=1}^{i} p_X(k) \geq \sum_{k=1}^{i} p_Y(k)$$
- Let $\mathbf{P}, \mathbf{P}'$ be two sub-stochastic matrices over $\mathcal{S}$. Then:
$$\mathbf{P} \leq_{st} \mathbf{P}' \text{ if } \forall i \ \ \mathbf{P}[s_i, -] \leq_{st} \mathbf{P}'[s_i, -]$$
- Let $\mathbf{P}$ be a sub-stochastic matrix over $\mathcal{S}$. Then:
$$\mathbf{P} \text{ is } st\text{-monotone if } \forall i < n \ \ \mathbf{P}[s_i, -] \leq_{st} \mathbf{P}[s_{i+1}, -]$$

The survival toolkit: some results

- Let X, Y be two defective distributions over $\mathcal{S}$ such that $X \leq_{st} Y$ and r be a decreasing mapping over $\mathcal{S}$. Then: $E(r(X)) \geq E(r(Y))$
- Let $\mathbf{P} \leq_{st} \mathbf{P}'$ be two sub-stochastic matrices over $\mathcal{S}$ such that either $\mathbf{P}$ or $\mathbf{P}'$ is st-monotone. Then:
  1. The inequality holds for every power of matrices: $\forall k \in \mathbb{N}\ \mathbf{P}^k \leq_{st} \mathbf{P}'^k$
  2. (as a corollary) the mean leaving time of $\mathbf{P}$ is greater than the one of $\mathbf{P}'$:
$$(\sum_{k \in \mathbb{N}} \mathbf{P}^k)\mathbf{1}_n \geq_{el} (\sum_{k \in \mathbb{N}} \mathbf{P}'^k)\mathbf{1}_n$$

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Stochastic bounds and IMCs

Motivation

How to compute (accurate) bounds for leaving time $\mathbf{m}[s]$ and $\mathbf{M}[s]$?

$$\mathbf{m}[s] \leq \min_{\mathbf{P}\in\mathcal{M}(\mathbf{P}^-,\mathbf{P}^+,\mathbf{out})}\{(\sum_{k\in\mathbb{N}}\mathbf{P}^k)\mathbf{1}_n)[s]\}$$

$$\max_{\mathbf{P}\in\mathcal{M}(\mathbf{P}^-,\mathbf{P}^+,\mathbf{out})}\{(\sum_{k\in\mathbb{N}}\mathbf{P}^k)\mathbf{1}_n)[s]\} \leq \mathbf{M}[s]$$

Computing the best $\mathbf{m}[s]$ is straightforward.

$$\min_{\mathbf{P}\in\mathcal{M}(\mathbf{P}^-,\mathbf{P}^+,\mathbf{out})}\{(\sum_{k\in\mathbb{N}}\mathbf{P}^k)\mathbf{1}_n)[s]\} = (\sum_{k\in\mathbb{N}}(\mathbf{P}^-)^k)\mathbf{1}_n)[s]$$

Computing a bound $\mathbf{M}[s]$ via stochastic order *(Haddad, Moreaux EJOR 2007)*

- There is a unique greatest lower bound $\mathbf{P}^\bullet$ w.r.t. $\leq_{st}$ for $\mathcal{M}(\mathbf{P}^-,\mathbf{P}^+,\mathbf{out})$
- which admits a unique greatest *monotone* lower bounding matrix $\mathbf{P}^\star \leq_{st} \mathbf{P}^\bullet$.
- $\mathbf{M}[s] = (\sum_{k\in\mathbb{N}}(\mathbf{P}^\star)^k)\mathbf{1}_n)[s]$
  (different bounds are possible depending on the ordering of states)
- Furthermore a priori detecting states s for which $\mathbf{M}[s] = \infty$ can be performed in very efficient way without computing the strongly connected components of the underlying graph *(this paper)*.

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IMC model

2 PCTL

Efficient Model Checking PCTL for IMCs

Conclusion and perspectives

PCTL for MCs

Syntax

$\phi ::= true \mid a \mid \phi \wedge \phi \mid \neg\phi \mid \mathcal{P}_{\lhd p}(\mathcal{X}\phi) \mid \mathcal{P}_{\lhd p}(\phi_1\ \mathcal{U}^{[\alpha,\beta]}\phi_2) \mid \mathcal{D}_{\lhd r}(\phi)$

Semantic: path formulas

A path $\sigma \equiv s_0, s_1, \ldots$ is an infinite sequence of states of the Markov chain.

- $\sigma \models \mathcal{X}\phi$ iff $s_1 \models \phi$
- $\sigma \models \phi_1 \mathcal{U} \phi_2$ iff there exists i such that $s_i \models \phi_2$ and $\forall j < i\ s_j \models \phi_1$

Semantic: state formulas

- Threshold formulas
  based on $Prob^{\mathcal{M}}(s, \varphi)$ *the probability that a random path in* $\mathcal{M}$ *starting from* s *satisfies* φ

  $$s \models \mathcal{P}_{\lhd p}(\varphi) \text{ iff } Prob^{\mathcal{M}}(s, \varphi) \lhd p$$

- Duration formulas
  based on $\mathrm{E}^{\mathcal{M}}(FTime(s, \phi))$ *the mean of the first time that a random path in* $\mathcal{M}$ *starting from* s *satisfies* ϕ

  $$s \models \mathcal{D}_{\lhd r}(\phi) \text{ iff } \mathrm{E}^{\mathcal{M}}(FTime(s, \phi)) \lhd r$$

PCTL for MCs

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PCTL for IMCs

Exact semantic

1. $\forall \mathcal{M} \in \mathcal{M}(\mathbf{P}^-, \mathbf{P}^+)\ \mathcal{M}, s \models \phi$ *(always satisfied)*
2. $\forall \mathcal{M} \in \mathcal{M}(\mathbf{P}^-, \mathbf{P}^+)\ \mathcal{M}, s \models \neg\phi$ *(never satisfied)*
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Approximate semantic induced by a semi-decisional procedure

Six possible alternative information labels for s w.r.t. ϕ

- $s.\phi = \forall^+$ when it is known that case 1 holds.
- $s.\phi = \forall^-$ when it is known that case 2 holds.
- $s.\phi = \exists^{+-}$ when it is known that case 3 holds.
- $s.\phi = \exists^+$ when it is known that cases 1 or 3 hold.
- $s.\phi = \exists^-$ when it is known that cases 2 or 3 hold.
- $s.\phi = ?$ when no information has been obtained.

PCTL for IMCs

Exact semantic

1. $\forall \mathcal{M} \in \mathcal{M}(\mathbf{P}^-, \mathbf{P}^+)\ \mathcal{M}, s \models \phi$ *(always satisfied)*
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Outline

IMC model

PCTL

3. Efficient Model Checking PCTL for IMCs

Conclusion and perspectives

General principles

First step. Split the set of states depending on:

- the current label $(\forall^{+}, \ldots)$ to be assigned to states;
- the labels of states w.r.t. the sub-formulas occurring in the formula;
- the external path operator of the formula;
- the kind of comparison $\leq, \geq$.

Second step. Build one or more sub-stochastic matrices

- by (appropriately) ordering the states inside the subsets;
- and applying an algorithm for IMC to compute the coefficients (maximizing cumulative probabilities, st-monotone glb matrix, etc.).

Third step. Perform a standard computation for Markov chains.

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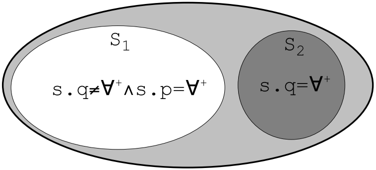
Assigning $\forall^-$ for formula $\mathcal{P}_{\leq p}(p\ \mathcal{U}^{[0,\beta]} q)$

First step.

The semi-decision procedure implies a conservative approach. Thus:

- $\mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2) = \{s \in \mathcal{S} \mid s.p \neq \forall^+ \wedge s.q \neq \forall^+\}$ is the set of states such that one cannot assign $\forall^-$.
  (the probability of satisfaction for the random path could be 0)
- $\mathcal{S}_2 = \{s \in \mathcal{S} \mid s.q = \forall^+\}$ is the set of states such that one can surely assign $\forall^-$.
  (the probability of satisfaction for the random path is 1)
- $\mathcal{S}_1 = \{s \in \mathcal{S} \mid s.p = \forall^+ \wedge s.q \neq \forall^+\}$ is the set of states that requires a (conservative) computation.



Assigning $\forall^-$ for formula $\mathcal{P}_{\leq p}(p\ \mathcal{U}^{[0,\beta]} q)$

Second step.
What is the quantity to lower bound?
The probability to reach $\mathcal{S}_2$ from $\mathcal{S}_1$ without leaving $\mathcal{S}_1$ in at most β steps:

$$\left(\sum_{k=0}^{\beta-1} (\mathbf{P}_{|\mathcal{S}_1 \times \mathcal{S}_1})^k\right) \cdot \mathbf{r}$$

where $\mathbf{r}[s]$ is the probability to immediately reach $\mathcal{S}_2$ from s.

So we perform the following substitutions:

- Matrice $\mathbf{P}^-$ is substituted to $\mathbf{P}$.
- Vector $\mathbf{r}$ is substituted by $\mathbf{r}^- = \max(\sum_{s' \in \mathcal{S}_2} \mathbf{P}^-[s,s'], 1 - \sum_{s' \notin \mathcal{S}_2} \mathbf{P}^+[s,s'])$

Third step. Compute $\boxed{\mathbf{m} = \left(\sum_{k=0}^{\beta-1} (\mathbf{P}^-_{|\mathcal{S}_1 \times \mathcal{S}_1})^k\right) \cdot \mathbf{r}^-}$
and assign $\forall^-$ to s iff $\mathbf{m}[s] > p$.

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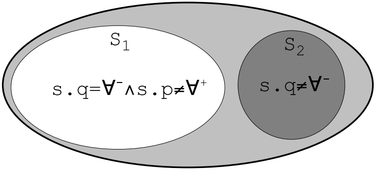

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Second step.
We now upper bound

$$\left(\sum_{k=0}^{\beta-1}(\mathbf{P}_{|\mathcal{S}_1\times\mathcal{S}_1})^k\right)\cdot\mathbf{r}$$

So we define an appropriate $\mathcal{M}(\mathbf{P}^-,\mathbf{P}^+,\mathbf{out})$ over $\mathcal{S}_1$:

- Matrices $\mathbf{P}^+,\mathbf{P}^-$ are the original matrices restricted to $\mathcal{S}_1$.
- Vector $\mathbf{out}$ is defined by:
  $\mathbf{out}[s]=\max(\sum_{s'\notin\mathcal{S}_1}\mathbf{P}^-[s,s'],1-\sum_{s'\in\mathcal{S}_1}\mathbf{P}^+[s,s'])$
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Warning In order to apply stochastic bound, $\mathbf{r}^+$ must be decreasing. So it implies a re-ordering of states of $\mathcal{S}_1$ before computing $\mathbf{P}^\star$.

Third step. Compute $\boxed{\mathbf{M}=\left(\sum_{k=0}^{\beta-1}(\mathbf{P}^\star)^k\right)\cdot\mathbf{r}^+}$

and assign $\forall^+$ to s iff $\mathbf{M}[s]\leq p$.

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Assigning $\exists^+$ for formula $\mathcal{P}_{\leq p}(p\ \mathcal{U}^{[0,\beta]} q)$

First step as in the previous case.

Second step.

Here we guess one (or more) matrix $\mathbf{P}$ with a small value of:

$$\left(\sum_{k=0}^{\beta-1}(\mathbf{P}_{|\mathcal{S}_1\times\mathcal{S}_1})^k\right)\cdot\mathbf{r}$$

- We order the states of $\mathcal{S}$: first $\mathcal{S}_2$ then $\mathcal{S}_1$ and $\mathcal{S}\setminus(\mathcal{S}_1\cup\mathcal{S}_2)$
- Inside S_1, order the states w.r.t. $\mathbf{r}[s]=\max(\sum_{s'\in\mathcal{S}_2}\mathbf{P}^+[s,s'],1-\sum_{s'\notin\mathcal{S}_2}\mathbf{P}^+[s,s'])$
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Outline

Conclusion and perspectives

Summary of results

- Efficient semi-decision procedure for model checking IMCs
- Application of stochastic comparisons for model checking PCTL over IMCs
- Handling the interval constrained until and the mean reachability time operators
- Providing partial answers $\exists^+$, $\exists^-$

Perspectives

- Development of a prototype for high level formalisms with IMC as possible semantic
- Accuracy of bounds and impact of heuristics
- One year post-doc position available for this project