Stochastic comparison for performability of telecommunication systems

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Motivation

- Performability evaluation of computer and telecommunication (more parameters so more complexe)
- Multidimensional Markov chains, state-space explosion problem
- Numerical analysis may be very complex or intractable

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- Performability evaluation of computer and telecommunication (more parameters so more complexe)
- Multidimensional Markov chains, state-space explosion problem
- Numerical analysis may be very complex or intractable
- Stochastic Comparison
  - Idea Find “simple models” to bound the considered performance measures
  - Method Application of stochastic comparison approach to construct bounding models

Stochastic Comparison on multidimensional state space

- State space E with $\preceq$
- Stochastic orderings : $\preceq_{st}, \preceq_{wk}, \preceq_{wk^*}$
  - $X \preceq_{st} Y \Leftrightarrow E(f(X)) \leq E(f(Y))$
  - $\{X(t), t \geq 0\} \preceq_{st} \{Y(t), t \geq 0\} \Leftrightarrow E(f(X(t)) \leq E(f(Y(t)), \forall t \geq 0$
  - $\{X(t), t \geq 0\} \preceq_{st} \{Y(t), t \geq 0\} \Rightarrow \sum_{x \in E} \Pi^X(x) f(x) \leq \sum_{x \in E} \Pi^Y(x) f(x)$

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- Different methods : coupling, increasing set $\Rrightarrow$ transition rates comparison
- Comparison by mapping functions : easier process (reduced state space or product form)

Comparison of the performance measures

For the Performability measure of the CTMC

$$R(t) = \sum_{x \in E} \Pi(x,t) f(x)$$

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$$R(t) = \sum_{x \in E} \Pi(x,t) f(x)$$

From the stochastic ordering relation we have :

For the upper bound

$$R(t) \leq R^u(t) = \sum_{x \in E^u} \Pi^u(x,t) f(x)$$

For the lower bound

$$R(t) \geq R^l(t) = \sum_{x \in E^l} \Pi^l(x,t) f(x)$$

Outline

Application : telecommunication system

Figure: Erlang Loss with performability Model, n=3

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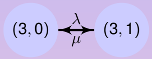

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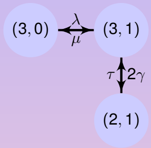

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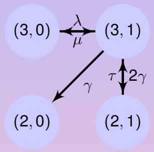

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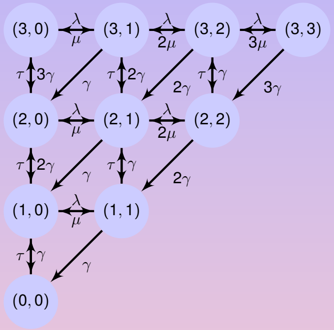

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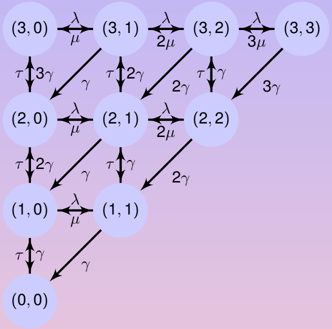
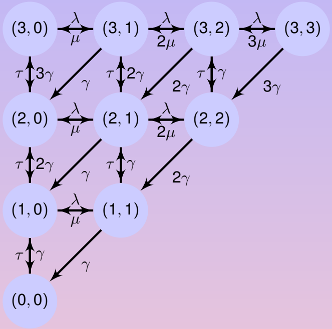


Figure: Erlang Loss with performability Model, n=3

- CTMC, $\{X(t),\ t \geq 0\}$ with $(n+1)(n+2)/2$ states and , $\forall x = (x_1, x_2) \in A$, $x_1 \geq x_2$.

Application : telecommunication system



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- CTMC, $\{X(t),\ t \geq 0\}$ with $(n+1)(n+2)/2$ states and , $\forall x = (x_1, x_2) \in A$, $x_1 \geq x_2$.
- Performance study : $T_b = \sum_{x_i=0}^{n} \Pi[x_i, x_i]$

Outline

Markov Reward Model contruction

MRM components: Pure Availability Model

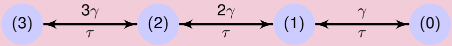

Figure: Pure Availability Model

$$\bar{A} = \pi_0 = [\sum_{i=0}^{n} \frac{1}{i!}(\tau/\gamma)^i]^{-1} \qquad \pi_i = ((\tau/\gamma)^i/i!)\pi_0 \tag{1}$$

Markov Reward Model contruction

MRM components: Pure Performance Model

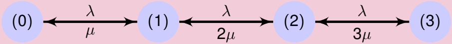

Figure: Pure Performance Model

$$p_b(i) = \frac{(\lambda/\mu)^i/i!)}{\sum_{j=0}^{i}(\lambda/\mu)^j/j!} \tag{2}$$

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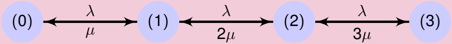

Figure: Pure Performance Model

$$p_b(i) = \frac{(\lambda/\mu)^i/i!)}{\sum_{j=0}^{i}(\lambda/\mu)^j/j!} \quad (2)$$

The total call blocking probability

$$T_b^* = \sum_{i=0}^{n} r_i \pi_i = \sum_{i=1}^{n} p_b(i)\pi_i + \pi_0 \quad (3)$$

Outline

System with a product form

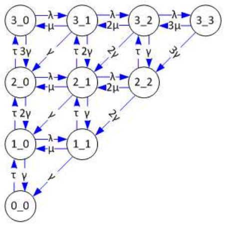

Figure: Exact model, X(t)

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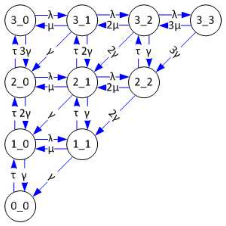

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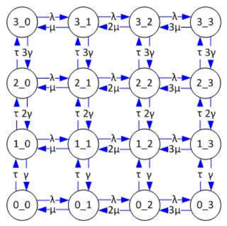

Figure: Product form model, $X^{sup1}(t)$

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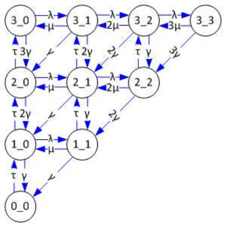

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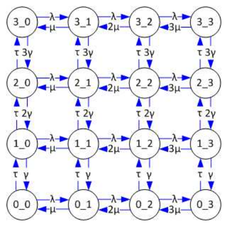

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Performability measure

- Total blocking probability $T_b = \sum_{x_i=0}^{n} \Pi[x_i, x_i]$

System with product form

Definition

- Many to one mapping function $g : E \rightarrow A$.

$$g(y_1, y_2) = \begin{cases} (y_1, y_1) & \text{if} \quad y_1 \leq y_2 \\ (y_1, y_2) & \text{otherwise} \end{cases} \tag{4}$$

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- We define on A the following order $\preceq$:

$$\forall x, y \in A, x \preceq y \iff x_1 - x_2 \geq y_1 - y_2 \tag{5}$$

Proofs: Coupling method

- Proposition:

$$\{X(t), t \geq 0\} \preceq_{st} \{g(X^{sup1}(t)), t \geq 0\} \tag{6}$$

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Theorem 1 (Stoyan, Doisy)

$$\{X(t), t \geq 0\} \preceq_{st} \{g(Y(t)), t \geq 0\} \tag{7}$$

if there exists the coupling $\{(\widehat{X}(t), \widehat{Y}(t)),\ t \geq 0\}$ such that:

$$\widehat{X}(0) \preceq g(\widehat{Y}(0)) \Rightarrow \widehat{X}(t) \preceq g(\widehat{Y}(t)),\ \forall t > 0 \tag{8}$$

Proofs

- Assumption : $\widehat{X}(t) \leq g(\widehat{X}^{sup1}(t))$
- Let see if $\widehat{X}(t+\Delta t) \leq g(\widehat{X}^{sup1}(t+\Delta t))$

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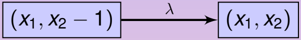

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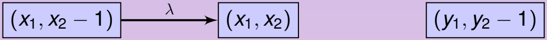

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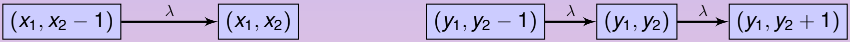

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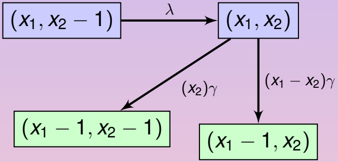

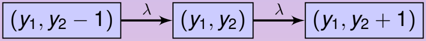

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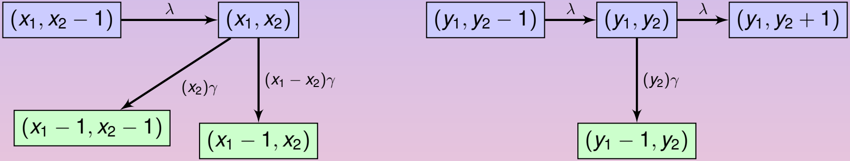

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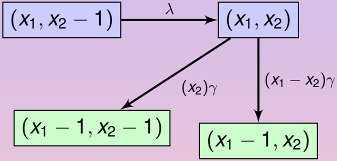

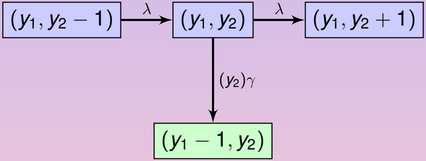

- We can deduce that the proposition $\{X(t), t \geq 0\} \preceq_{st} \{g(X^{sup1}(t)), t \geq 0\}$ is proved.

Proofs

- We have:

$$\sum_{z \in \Gamma} \Pi(x) \leq \sum_{g(x) \in \Gamma} \Pi^{sup1}(x), \forall \Gamma \in \Phi_{st}(A) \quad (9)$$

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- Let $\Gamma = \{(0,0),(1,1),\ldots(n,n)\}$ an increasing set of $\Phi_{st}(A)$, then from equation **??** we have :

$$\sum_{x \mid x_1 = x_2} \Pi(x) \leq \sum_{x \in E \mid x_1 \leq x_2} \Pi^{sup1}(x). \tag{10}$$

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- So we obtain :

$$T_b \leq T_b^{sup1} = \sum_{x \in E \mid x_1 \leq x_2} \Pi^{sup1}(x). \tag{11}$$

Outline

Bounding aggregation definition

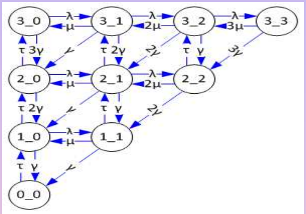


Figure: Exact model

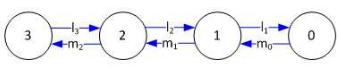
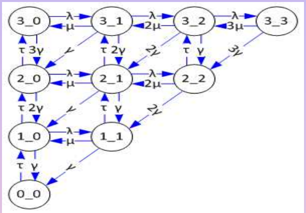


Figure: Birth and death model (Upper and Lower)

Bounding aggregation definition



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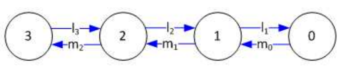


Figure: Birth and death model (Upper and Lower)

- Proof the monotonicity of the Birth and Death process with rates state's dependence
- Define bounding systems according rates definition

Bounding aggregation definition

Definition

- Many to one mapping function $h : A \to F$.

$$\forall x = (x_1, x_2) \in A, h(x) = x_1 - x_2 \tag{12}$$

- We define on F the following order $\preceq$:

$$\forall x_1, x_2 \in F, x_1 \preceq x_2 \iff x_1 \geq x_2 \tag{13}$$

Proof

Theorem 2 (Massey, Stoyan)

If the following conditions 1, 2, 3 are satisfied:

1. $g(X(0)) \preceq_{st} Y(0)$
2. $Y(t)$ is $\preceq_{st}$-monotone
3. $\sum_{g(z)\in\Gamma} Q_1(y,z) \leq \sum_{z\in\Gamma} Q_2(x,z), \ \forall\Gamma \in \Phi_{st}(F),$
   $\forall x \in S, \ y \in A \mid g(y) = x$

then we have:

$$\{g(X(t)), t \geq 0\} \preceq_{st} \{Y(t), t \geq 0\}$$

Proof: Monotonicity

Theorem 3 (Massey, Stoyan)

$\{X(t), t \geq 0\}$ is $\preceq_{st}$ -monotone if and only if $\forall \Gamma \in \Phi_{st}(A)$,

$$\sum_{z \in \Gamma} Q(x,z) \leq \sum_{z \in \Gamma} Q(y,z), \forall x \preceq y \mid x, y \in \Gamma, \ or \ x, y \notin \Gamma$$

Monotonicity proof

$$\sum_{z \in \Gamma} Q_2(x,z) \leq \sum_{z \in \Gamma} Q_2(y,z) \tag{14}$$

For all states $x, y \in F$, and all increasing sets $\Gamma \in \Phi_{st}(F)$ such that :

$$\forall x \preceq y \in F \mid x, y \in \Gamma \ or \ x, y \notin \Gamma \tag{15}$$

Proof: Monotonicity

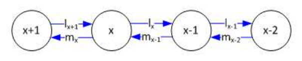


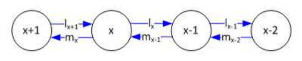
Figure: Aggregated Model

We define the two increasing set:

$\Gamma_x = \{x, \dots, y, \dots\}$; $\Gamma_{y-1} = \{y-1, \dots\}$

Γ_x	Γ_{y-1}

Proof: Monotonicity



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Γ_x	Γ_{y-1}
$\sum_{z \in \Gamma_x} Q_2(x,z) = -m_x$ $\sum_{z \in \Gamma_x} Q_2(y,z) = 0$	$\sum_{z \in \Gamma_{y-1}} Q_2(x,z) = 0$ $\sum_{z \in \Gamma_{y-1}} Q_2(y,z) = l_y$

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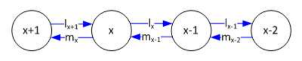

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$\sum_{z\in\Gamma_x} Q_2(y,z) = 0$	$\sum_{z\in\Gamma_{y-1}} Q_2(y,z) = l_y$

- So the inequality $\sum_{z\in\Gamma} Q_2(x,z) \leq \sum_{z\in\Gamma} Q_2(y,z)$ is verified, and proposition $Y(t)$ is $\preceq_{st}$-monotone is proved.

Proof

- We denote: $Y^{sup2}(t)$, the birth death process defined so as an Upper Bound
- For any state x, we have:

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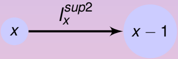

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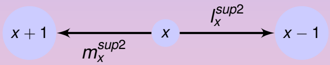

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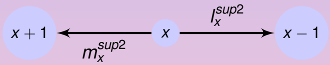

- $x \in F$ for $Y^{sup2}(t)$, transition rates will be different from 0 for the increasing sets Γ_{x-1}, and Γ_x

Proof

Upper bound system definition

Γ_{x-1}	Γ_x
$\sum_{h(z)\in\Gamma_{x-1}} Q(y,z) \leq l_x^{sup2}, \forall y \in A \mid h(y) = x$ $l_x^{sup2} = max_{y \mid h(y)=x} \sum_{h(z)\in\Gamma_{x-1}} Q(y,z)$ $l_x^{sup2} = \lambda + x\gamma$	$\sum_{h(z)\in\Gamma_x} Q(y,z) \geq m_x^{sup2}, \forall y \in A \mid h(y) = x$ $m_x^{sup2} = min_{y \mid h(y)=x} \sum_{h(z)\in\Gamma_{x-1}} Q(y,z)$ $m_x^{sup2} = min(\mu, \tau)$
We compute	
π^{sup2}	
and we derive	
$T_b^{sup2} = \pi^{sup2}(0)$	
we obtain	
$T_b = \sum_{x_1=0}^{n} \Pi[x_1, x_1] \leq= T_b^{sup2}$	

Proof

Lower bound system definition

Γ_{x-1}	Γ_x
$\sum_{h(z)\in\Gamma_{x-1}} Q(y,z) \geq l_x^{inf2}, \forall y \in A \mid h(y) = x$ $l_x^{inf2} = min_{y \mid h(y)=x} \sum_{h(z)\in\Gamma_{x-1}} Q(y,z)$ $l_x^{inf2} = \lambda + x\gamma$	$\sum_{h(z)\in\Gamma_x} Q(y,z) \leq m_x^{inf2}, \forall y \in A \mid h(y) = x$ $m_x^{inf2} = max_{y \mid h(y)=x} \sum_{h(z)\in\Gamma_{x-1}} Q(y,z)$ $m_x^{inf2} = \tau + (n-x)\mu$
We compute	
π^{inf2}	
and we derive	
$T_b^{inf2} = \pi^{inf2}(0)$	
we obtain	
$T_b = \sum_{x_1=0}^{n} \Pi[x_1, x_1] \geq= T_b^{inf2}$	

Outline

Numerical results

Definition

- n=10 and n=27 channels
- Call arrival rate varing from 2 to 50 calls per min
- Channel failure occurs every hour or 10 hours.
- We use SHARPE package for computation.

Blocking probability analysis

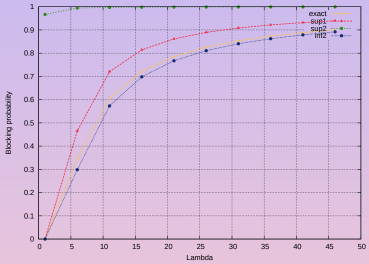

Figure: $1/\mu = 2min, \gamma = 1/hours, 1/\tau = 30min, n = 10$

Blocking probability analysis

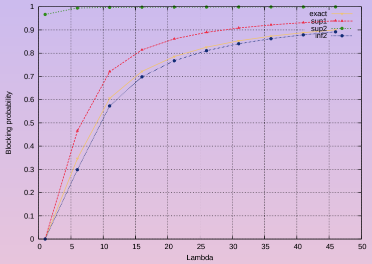

Figure: $1/\mu = 2min, \gamma = 1/hours, 1/\tau = 30min, n = 10$

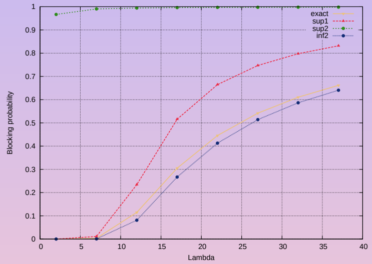

Figure: $1/\mu = 2mn, \gamma = 1/10hours, 1/\tau = 30min, n = 27$

Blocking probability analysis

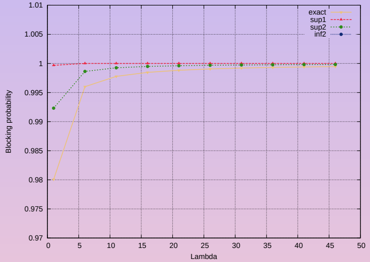

Figure: $1/\mu = 6min, \gamma = 5/hours, 1/\tau = 2hours, n = 10$

Remark

- Parameters have an impact on the quality of bounds
- When the failure rate are low, the lower bound is intersesting and sup1 is better than sup2
- When the failure rate are high, sup2 is better than sup1.

Conclusion

Conclusion

- The Trivedi approximation is an upper bound
- Ordering not usually easier to establish
- Differents bounds for the original system
- Generalize to several groups of channels.