Bounds quality for performance evaluation of computer networks [1]

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- **Problem :**
  - On multidimensional state space, different stochastic orderings
  - Quality of bounding systems ?
  - Which ordering provides the best bounding systems ?
- **Proposition :** We study a system represented by a multidimensionnal Markov process with no product form $\Rightarrow$ Different bounding systems, and comparison from performance measure

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Stochastic ordering (Increasing sets)

E a state space with a preorder $\preceq$ (reflexive, transitive)

$$X \preceq_{\Phi} Y \Leftrightarrow P(X \in \Gamma) \leq P(Y \in \Gamma), \; \forall \Gamma \in \Phi(E)$$

$$\Phi_{st}(E) = \{\text{all increasing sets on } E\}$$

$$\Phi_{wk}(E) = \{\{x\} \uparrow, x \in E\} \cup E$$
$$\{x\} \uparrow = \{y \in E | y \succeq x\}$$

and

$$\Phi_{wk^*}(E) = \{E - \{x\} \downarrow, x \in E\} \cup E, \; where \; \{x\} \downarrow = \{y \in E | y \preceq x\}$$

$\Phi_{st}(E) \rightarrow \preceq_{st}$, $\Phi_{wk}(E) \rightarrow \preceq_{wk}$, $\Phi_{wk^*}(E) \rightarrow \preceq_{wk^*}$ stochastic orderings.
$\Phi_{wk}(E) \subset \Phi_{st}(E)$, and $\Phi_{wk^*}(E) \subset \Phi_{st}(E)$.

Example

$$E = \{(0,0),(0,1),(1,0),(1,1)\}$$

$$\Phi_{wk}(E) = \{E, \{(0,1),(1,1)\}, \{(1,0),(1,1)\}, \{(1,1)\}\}$$

$$\Phi_{st}(E) = \Phi_{wk}(E) \cup \{(0,1),(1,0),(1,1)\}$$

$$P_X = (0.4, 0.2, 0.2, 0.2), P_Y = (0.5, 0.1, 0.1, 0.3), P_X \preceq_{wk} P_Y, \; P_X \not\preceq_{st} P_Y$$

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The system understudy

The system understudy is similar to a Jackson network except that queues have a finite capacity, each queue i has the following parameters :

- Finite capacity B_i
- Exponential Inter-arrival times with parameters λ_i. If the queue is not full the customer is accepted in the queue, otherwise it is lost.
- Exponential service times, with parameters μ_i, and after the service, we have :
  - with the probability p_{ij} the customer transits from the queue i to the queue j, if queue j is not full. Otherwise, the customer is lost.
  - with the probability d_i the customer goes out.

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Problem Understudy

- Computing loss probabilities on $X(t)$?
- We propose to define different bounding systems by creating independance between queues
  - Making capacities infinite : Jackson network ($S2(t)$) : coupling method, we will prove : $X(t) \preceq_{st} S2(t)$
  - Cutting links between queues : n Independent $M/M/1/B_i$ queues : W(t), Increasing sets, we will prove : $X(t) \preceq_{wk} W(t)$
- Quality of bounding systems from loss probabilities
- No relations between bounding systems : $W(t) \not\preceq_{st} S2(t)$, and $W(t) \not\preceq_{wk} S2(t)$

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Bounding system 1 : Independent $M/M/1/B_i$ queues

Bounding System 1 is represented by n queues, each queue i has the following assumptions

- Arrival rate : $\lambda_i + \sum_{k=1,k\neq i}^{n} \mu_k p_{ki}$
- Service rate μ_i

The evolution is represented by the Markov process $W(t)$
We will see that :

$$\{X(t), t \geq 0\} \not\preceq_{st} \{W(t), t \geq 0\}$$

and

$$\{X(t), t \geq 0\} \preceq_{wk} \{W(t), t \geq 0\}$$

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We use the coupling method : we suppose

$$\widehat{X}(t) \preceq \widehat{W}(t), \text{ we see} : \widehat{X}(t+\Delta t) \preceq \widehat{W}(t+\Delta t) \, ? \qquad (1)$$

1. An arrival in queue i in $X(t)$ is compensated by an arrival in $W(t)$: $\lambda_i \leq \lambda_i + \sum_{k=1,k\neq i}^{n} \mu_k p_{ki}$
2. A transit from queue i to queue j in $X(t)$ is compensated by an arrival in queue j in $W(t)$: $\mu_i p_{ji} < \lambda_j + \sum_{k=1,k\neq j}^{n} \mu_k p_{kj} - \lambda_j = \sum_{k=1,k\neq j}^{n} \mu_k p_{kj}$
3. A service in queue i in $W(t)$ is not compensated by a service in $X(t)$ as $\mu_i d_i \leq \mu_i$

And so we may have at time $t + \Delta t$:

$$\widehat{X}(t+\Delta t) \not\preceq \widehat{W}(t+\Delta t) \qquad (2)$$

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Since we want to define an upper bound to the process $X(t)$, we consider two solutions :

1. we propose to verify if : $X(t) \preceq_{wk} W(t)$,
2. we propose to modify $W(t)$ by defining another process $S1(t)$ which could represent an upper bounding system : $X(t) \preceq_{st} S1(t)$

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A simple strong bounding system

1. $\{S1(t), t \geq 0\}$ is a multidimensional Markov process representing the evolution of a queueing system with independent $M/M/1/B_i$ queues defined as follows.
2. Each queue i :
   1. arrival rates $\lambda_i + \sum_{k=1,k\neq i}^{n} \mu_k p_{ki}$,
   2. service rate $\mu_i d_i$.

So we can deduce from the coupling method the following proposition :

$$\{X(t), t \geq 0\} \preceq_{st} \{S1(t), t \geq 0\} \tag{3}$$

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We have to prove :

$$\{X(t), t \geq 0\} \preceq_{wk} \{W(t), t \geq 0\} \tag{4}$$

We use the following theorem :

$$\{X(t), t \geq 0\} \preceq_{\Phi} \{Y(t), t \geq 0\} \tag{5}$$

if and only if the following conditions are verified :

1. $X(0) \preceq_{\Phi} Y(0)$
2. $\{X(t), t \geq 0\}$ or $\{Y(t), t \geq 0\}$ is $\preceq_{\Phi}$-monotone
3. $$\forall x \in E \sum_{z \in \Gamma} A(x,z) \leq \sum_{z \in \Gamma} B(x,z), \ \forall \Gamma \in \Phi(E) \tag{6}$$

From Massey :

Theorem

$\{X(t), t \geq 0\}$ is $\preceq_{st}$-monotone (increasing)if the following condition is verified :

$$\forall \Gamma \in \Phi_{st}(E),\ \forall x \preceq y \in E$$

$$\sum_{z \in \Gamma} A(x,z) \leq \sum_{z \in \Gamma} A(y,z),\ \ x, y \in \Gamma \ or \ x, y \notin \Gamma \qquad (7)$$

$\preceq_{wk}$- monotonicity

Theorem

$\{X(t), t \geq 0\}$ is $\preceq_{\Phi}$-monotone (increasing)if the following condition is verified :

$$\forall \Gamma \in \Phi(E),\ \forall x \preceq y \in E$$

$$\sum_{z \in \Gamma} A(x,z) \leq \sum_{z \in \Gamma} A(y,z),\ \ x, y \in \Gamma \text{ or } x, y \notin \Gamma \tag{8}$$

Proved in : "Stochastic monotonicity in queueing networks", H.Castel-Taleb, N.Pekergin, EPEW'09, 6th European Performance Engineering Workshop, Imperial College London, 9-10 July

First step : Definition of increasing sets

From events :

- arrival in queue i : $x \rightarrow x + e_i$
- service in queue i : $x \rightarrow x - e_i$
- transit from queue i to queue j : x $\rightarrow x - e_i + e_j$

As we must also take the condition :

$$x, y \in \Gamma \; or \; x, y \notin \Gamma$$

$$S_{wk}(E) = \{\{x\} \uparrow, \{x + e_i\} \uparrow, \{y + e_i\} \uparrow, \{x - e_i\} \uparrow, \{y - e_i\} \uparrow\}$$

If $x_i < B_i$:

$$\{x + e_i\} \uparrow = \{x + e_i, \ldots, y + e_i, \ldots\}$$

If $y_i < B_i$: :

$$\{y + e_i\} \uparrow = \{y + e_i, \ldots.\}$$

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If $x_i < B_i$:

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If $y_i < B_i$: :

$$\{y + e_i\} \uparrow = \{y + e_i, ...\}$$

Second step : transition rates comparisons

Γ	$\sum_{z\in\Gamma} Q^W(x,z)$	$\sum_{z\in\Gamma} Q^W(y,z)$
Γ_{x+e_i}	$\lambda_i + \sum_{k=1,k\neq i}^n \mu_k p_{ki}$	$\lambda_i + \sum_{k=1,k\neq i}^n \mu_k p_{ki}$
Γ_{y+e_i}	0	$\lambda_i + \sum_{k=1,k\neq i}^n \mu_k p_{ki}$
Γ_x	$-\sum_{k=1}^n \mu_k 1_{x_k>0}$	$-\sum_{k=1}^n \mu_k 1_{y_k>0} 1_{y_k=x_k}$
Γ_{x-e_i}	$-\sum_{k=1,k\neq i}^n \mu_k 1_{x_k>0}$	$-\sum_{k=1,k\neq i}^n \mu_k 1_{y_k>0} 1_{y_k=x_k}$
Γ_{y-e_i}	$-\sum_{k=1}^n \mu_k 1_{x_k>0}$	$-\sum_{k=1,k\neq i}^n \mu_k 1_{y_k>0}$

$\Gamma_{x+e_i} = \{x+e_i\}\uparrow$, $\Gamma_x = \{x\}\uparrow$, $\Gamma_{x-e_i} = \{x-e_i\}\uparrow$,
$\Gamma_{y+e_i} = \{y+e_i\}\uparrow$, $\Gamma_{y-e_i} = \{x-e_i\}\uparrow$.

$$\forall \Gamma \in S_{wk}(E),\ \sum_{z\in\Gamma} Q^W(x,z) \leq \sum_{z\in\Gamma} Q^W(y,z)$$

$$\forall x \preceq y \mid x,y \in \Gamma,\ or\ x,y \notin \Gamma$$

Generators comparisons : Q **and** Q^W $\Gamma_{x+e_i}, \Gamma_{x-e_j+e_i}, \Gamma_x, \Gamma_{x-e_i}$.

Γ	$\sum_{z\in\Gamma} Q(x,z)$	$\sum_{z\in\Gamma} Q^W(x,z)$
Γ_{x+e_i}	λ_i	$\lambda_i + \sum_{k=1,k\neq i}^{n} \mu_k p_{ki}$
$\Gamma_{x-e_j+e_i}$	$\mu_j p_{ji} + \lambda_i$	$\lambda_i + \sum_{k=1,k\neq i}^{n} \mu_k p_{ki}$
Γ_x	$-\sum_{k=1}^{n} \mu_k 1_{x_k>0}$	$-\sum_{k=1}^{n} \mu_k 1_{x_k>0}$
Γ_{x-e_i}	$-\sum_{k=1,k\neq i}^{n} \mu_k 1_{x_k>0}$	$-\sum_{k=1,k\neq i}^{n} \mu_k 1_{x_k>0}$

$$\forall \Gamma \in S_{wk}(E), \sum_{z\in\Gamma} Q^W(x,z) \leq \sum_{z\in\Gamma} Q^W(y,z)$$

$$\forall x \preceq y \mid x, y \in \Gamma, \ or \ x, y \notin \Gamma$$

$$P(X(t) \in \Gamma) \leq P(W(t) \in \Gamma), \forall \Gamma \in \Phi_{wk}(E) \tag{9}$$

and so for the stationary probability distributions we have :

$$\sum_{x \in \Gamma} \Pi(x) \leq \sum_{x \in \Gamma} \Pi^W(x), \forall \Gamma \in \Phi_{wk}(E) \tag{10}$$

$$\text{Loss probability } LX_i = \sum_{x \in E | x_i = B_i} \Pi(x)$$

Let $x^* = (0, \ldots, B_i, \ldots 0)$, and $\Gamma = \{x*\} \uparrow \in \Phi_{wk}(E)$.

$$LX_i = \sum_{x \in \Gamma} \Pi(x)$$

As $\Gamma = \{x*\} \uparrow \in \Phi_{wk}(E)$,

$$LX_i \leq LW_i \; LW_i = \sum_{x \in \Gamma} \Pi^W(x) \tag{11}$$

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Comparison of finite process with an infinite process

$$\{X(t), t \geq 0\} \preceq_{st} \{S2(t), t \geq 0\} \quad (12)$$

$$\textit{Suppose}: \ \widehat{X}(t) \preceq \widehat{S2}(t); \ \textit{and show}: \widehat{X}(t+\Delta t) \preceq \widehat{S2}(t+\Delta t) \quad (13)$$

1. An arrival in queue i in $X(t)$ is compensated by an arrival in queue$i$ in $S2(t)$ (same arrival rate λ_i). No arrival if queue i is full in $X(t)$ and an arrival in $S2(t)$
2. A transit from queue i to queue j in $X(t)$ is compensated by the same event in $S2(t)$ (same rate $\mu_i p_{ij}$). If queue j is full in $X(t)$ then $X_i(t)$ decreases (the customer goes out), and in $S2(t)$ there is the transit.
3. a service from queue i in $X(t)$ is compensated by the same event in $S2(t)$ (the service rate is $\mu_i d_i$)

$$\{X(t), t \geq 0\} \preceq_{st} \{S2(t), t \geq 0\} \tag{14}$$

If the stability condition is satisfied, then the stationary probability distribution Π^{S2} exists. So we have the following inequality :

$$\sum_{x \in \Gamma} \Pi(x) \leq \sum_{x \in \Gamma} \Pi^{S2}(x), \forall \Gamma \in \Phi_{st}(E) \tag{15}$$

The exact loss probability LX_i on queue i for the process $\{X(t), t \geq 0\}$ is given by the following formula :

$$LX_i = \sum_{x \succeq x^*} \Pi(x) \qquad (16)$$

So we propose to compute different loss probabilities bounds for each queue i :

- The weak bound LW_i on the process $W(t)$ generated by the weak ordering.
- The Strong1 bound $LS1_i$ on the process $S1(t)$, which represents a simple bound
- The Strong2 bound $LS2_i$ on the process $S2(t)$ which represents a more refined bound.

The goal is to compare LW_i, $LS1_i$, and $LS2_i$

$$\{X(t), t \geq 0\} \preceq_{wk} \{W(t), t \geq 0\} \tag{17}$$

we have for $\Gamma = \{x \succeq x*\} \in \Phi_{wk}(E)$:

$$LX_i \leq LW_i \ LW_i = \sum_{x \succeq x^*} \Pi^W(x) \tag{18}$$

As $\Gamma = \{x \succeq x*\} \in \Phi_{st}(E)$:

$$LX_i \leq LS1_i \ LS1_i = \sum_{x \succeq x^*} \Pi^{S1}(x) \tag{19}$$

$$LX_i \leq LS2_i, \ LS2_i = \sum_{x \succeq x^*} \Pi^{S2}(x) \tag{20}$$

Loss probabilities bounds on independent $M/M/1/B_i$ *queues*

The loss probability LW_i is computed from the weak bound $\{W(t), t \geq 0\}$, and is equivalent to the loss probability in an $M/M/1/B_i$ queue :

$$LW_i = a_i^{B_i} \frac{1 - a_i}{1 - a_i^{B_i+1}}, \ \textit{where } a_i = \frac{\lambda_i + \sum_{k=1, k\neq i}^{n} \mu_k p_{ki}}{\mu_i} \qquad (21)$$

$$LS1_i = b_i^{B_i} \frac{(1 - b_i)}{1 - b_i^{B_i+1}}, \ \textit{where } b_i = \frac{\lambda_i + \sum_{k=1, k\neq i}^{n} \mu_k p_{ki}}{\mu_i d_i} \qquad (22)$$

we have :

$$LW_i \leq LS1_i$$

Loss probabilities on infinite capacity queues in Jackson network

$$LS2_i = \sum_{x_i=B_i}^{\infty} c_i^{x_i}(1-c_i), \ \textit{where } c_i = \frac{\Lambda_i}{\mu_i} \tag{23}$$

$$\Lambda_i = \lambda_i + \sum_{k=1,k\neq i}^{n} \Lambda_k p_{ki}, c_i < 1$$

As

$$\sum_{x_i=0}^{\infty} c_i^{x_i}(1-c_i) = 1$$

$$\sum_{x_i=0}^{Bi-1} c_i^{x_i} = \frac{1-c_i^{B_i}}{1-c_i}, \textit{then we obtain} \sum_{x_i=B_i}^{\infty} c_i^{x_i} = \frac{c_i^{B_i}}{1-c_i} \tag{24}$$

$$LS2_i = c_i^{B_i} \tag{25}$$

We know that : $LW_i \leq LS1_i$,
What is the relation between :

$$LW_i = a_i^{B_i} \frac{1 - a_i}{1 - a_i^{B_i+1}} \tag{26}$$

and :

$$LS2_i = c_i^{B_i}, \; c_i = \frac{\Lambda_i}{\mu_i} \tag{27}$$

It is clear that :

$$c_i < a_i, \; then \; c_i^{B_i} < a_i^{B_i}$$

but as

$$\frac{1 - a_i}{1 - a_i^{B_i+1}} < 1, \; then \; LW_i \not\leq LS2_i$$

Numerical example

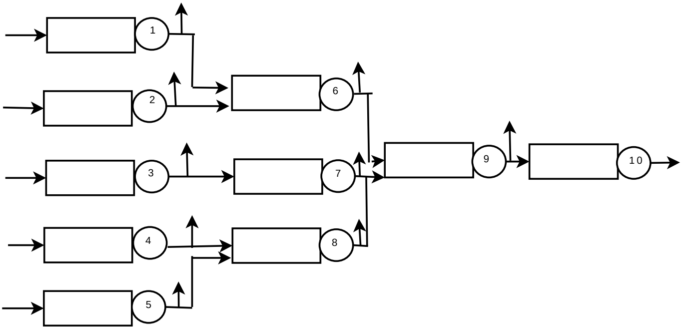

FIGURE: Queueing system understudy

Loss probabilities in queue 9

Input parameters

Queue : i	λ_i	μ_i	d_i	p_{ij}
1	168	170	0.2	0.8
2	40	41	0.2	0.8
3	110	112	0.2	0.8
4	82	84	0.2	0.8
5	82	84	0.2	0.8
6	0	170	0.1	0.9
7	0	91	0.1	0.9
8	0	136	0.1	0.9
9	0	480	0.8	0.2
10	0	500	1	0

TABLE: Input parameters values

Weak and Strong1 bounds

$a_9 = 0.743$, and $b_9 = 0.929$.

$$a_i = \frac{\lambda_i + \sum_{k=1,k\neq i}^{n} \mu_k p_{ki}}{\mu_i}$$

$$b_i = \frac{\lambda_i + \sum_{k=1,k\neq i}^{n} \mu_k p_{ki}}{\mu_i d_i}$$

B_9	LW_9 (Weak)	$LS1_9$ (Strong1)
20	$6.887 * 10^{-4}$	0.0208
30	$3.560 * 10^{-5}$	0.0088
40	$1.8439 * 10^{-6}$	0.004
50	$9.5501 * 10^{-6}$	0.0018
60	$4.9463 * 10^{-9}$	$8.9612 * 10^{-4}$
70	$2.5618 * 10^{-10}$	$4.2961 * 10^{-4}$
80	$1.326 * 10^{-11}$	$2.06622 * 10^{-4}$
90	$6.87 * 10^{-13}$	$9.9521 * 10^{-5}$
100	$3.559 * 10^{-14}$	$4.7974 * 10^{-5}$

The strong1 bound for different values of d_9

B_9	$d_9 = 0.85$	$d_9 = 0.9$
20	0.009	0.003
30	0.002	$5.82 * 10^{-4}$
40	$6.18 * 10^{-4}$	$8.71 * 10^{-5}$
50	$1.6 * 10^{-4}$	$1.3041 * 10^{-5}$
60	$4.333 * 10^{-5}$	$1.95 * 10^{-6}$
70	$1.14 * 10^{-5}$	$2.92 * 10^{7}$
80	$3.04 * 10^{-6}$	$4.38 * 10^{8}$
90	$8.08 * 10^{-7}$	$6.56 * 10^{-9}$
100	$2.14 * 10^{-7}$	$9.83 * 10^{-10}$

TABLE: Strong1 bound $LS1_9$ for different values of d_9

$a_9 = 0.743$, and $b_9 = 0.929$, and $c_9 = 0.722$.

B_9	$LW_9(Weak)$	$LS1_9(Strong1)$	$LS2_9(Strong2)$
20	$6.887 * 10^{-4}$	0.0208	0.0015
30	$3.560 * 10^{-5}$	0.0088	$5.9244 * 10^{-5}$
40	$1.8439 * 10^{-6}$	0.004	$2.3095 * 10^{-6}$
50	$9.5501 * 10^{-8}$	0.0018	$9.0035 * 10^{-8}$
60	$4.9463 * 10^{-9}$	$8.9612 * 10^{-4}$	$3.5098 * 10^{-9}$
70	$2.5618 * 10^{-10}$	$4.2961 * 10^{4}$	$1.3682 * 10^{-10}$
80	$1.326 * 10^{-11}$	$2.06622 * 10^{4}$	$5.3340 * 10^{-12}$
90	$6.87 * 10^{-13}$	$9.9521 * 10^{-5}$	$2.0794 * 10^{-13}$
100	$3.559 * 10^{-14}$	$4.7974 * 10^{-5}$	$8.106 * 10^{-15}$

TABLE: Weak, Strong1 and Strong2 bounds

$\mu_9 = 360,\ a_9 = 0.991,\ c_9 = 0.9638$

B_9	$LW_9(Weak)$	$LS2_9(Strong2)$
20	0.043	0.479
30	0.028	0.331
40	0.020	0.229
50	0.0157	0.158
60	0.0126	0.11
70	0.010	0.076
80	0.0086	0.052
90	0.00736	0.0365
100	0.006	0.025

TABLE: Weak and Strong2 bounds for $c_9 = 0.9638$ and $a_9 = 0.991$

$\mu_9 = 500$, $c_9 = 0.694$, and $a_9 = 0.714$,

B_9	$LW_9(Weak)$	$LS2_9(Strong2)$
20	$3.3938 * 10^{-4}$	$6.717 * 10^{-4}$
30	$1.1676 * 10^{-5}$	$1.7409 * 10^{-5}$
40	$4.0206 * 10^{-7}$	$4.5121 * 10^{-7}$
50	$1.384 * 10^{-8}$	$1.1694 * 10^{-8}$
60	$4.76714 * 10^{-10}$	$3.0308 * 10^{-10}$
70	$1.641 * 10^{-11}$	$7.8553 * 10^{-12}$
80	$5.6522 * 10^{-13}$	$2.0359 * 10^{-13}$
90	$1.9462 * 10^{-14}$	$5.2765 * 10^{-15}$
100	$6.7012 * 10^{-16}$	$1.3675 * 10^{-16}$

TABLE: Weak and Strong2 bounds for $c_9 = 0.694$, $a_9 = 0.714$

$\mu_9 = 600$: $a_9 = 0.59$, and $c_9 = 0.57$.

B_9	$LW_9(Weak)$	$LS2_9(Strong2)$
20	$1.2525 * 10^{-5}$	$1.7521 * 10^{-5}$
30	$6.9656 * 10^{-8}$	$7.3340 * 10^{-8}$
40	$3.8737 * 10^{-10}$	$3.0699 * 10^{-10}$
50	$2.1542 * 10^{-12}$	$1.2850 * 10^{-12}$
60	$1.1980 * 10^{-14}$	$5.3788 * 10^{-15}$
70	$6.6626 * 10^{-17}$	$2.2515 * 10^{-17}$
80	$3.7052 * 10^{-19}$	$9.4244 * 10^{-20}$
90	$2.0605 * 10^{-21}$	$3.9449 * 10^{-22}$
100	$1.1459 * 10^{-23}$	$1.6512 * 10^{-24}$

TABLE: Weak and Strong2 bounds for $a_9 = 0.59$, $c_9 = 0.57$

Next, we modify the routing probabilities of queues 6,7 and 8 into queue 9. We take 0.8 instead of 0.9. We obtain $c_9 = 0.51$, and $a_9 = 0.52$.

B_9	$LW_9(Weak)$	$LS2_9(Strong2)$
20	$1.38 * 10^{-6}$	$1.674 * 10^{-6}$
30	$2.365 * 10^{-9}$	$2.165 * 10^{-9}$
40	$4.04 * 10^{-12}$	$2.80 * 10^{-12}$
50	$6.93 * 10^{-15}$	$3.62 * 10^{-15}$
60	$1.18 * 10^{-17}$	$4.69 * 10^{-18}$
70	$2.03 * 10^{-20}$	$6.07 * 10^{-21}$
80	$3.48 * 10^{-23}$	$7.85 * 10^{-24}$
90	$5.96 * 10^{-26}$	$1.01 * 10^{-26}$
100	$1.02 * 10^{-28}$	$1.31 * 10^{-29}$

TABLE: Weak and Strong2 bounds for $a_9 = 0.52$, $c_9 = 0.51$

Results Analysis

- When the load c_i is high, the Weak bound is better : the arrival rate of the weak bound $\lambda_i + \sum_{k=1,k\neq i}^{n} \mu_k p_{ki}$ is very close to the arrival rate of the Strong2 bound $\lambda_i + \sum_{k=1,k\neq i}^{n} \Lambda_k p_{ki}$, but the finite capacity is better than an infinite
- When the load c_i is low, the Strong2 bound is better especially for high capacities

Conclusion : Which bound is the best ?

	High load	Low load
High Capacity	The Weak	Strong2
Low Capacity	The Weak	The Weak