

# CUT : Combining stochastic ordering and censoring to bound steady-state rewards and first passage time

Jean-Michel Fourneau\*<sup>‡</sup>

\*INRIA project MESCAL,  
Montbonnot France  
Email : jmf@prism.uvsq.fr

Nihal Pekergin<sup>†‡</sup>

<sup>†</sup> Marin Mersenne Laboratory,  
Université Paris 1, Paris France  
Email : nih@prism.uvsq.fr

Sana Younès<sup>‡</sup>

<sup>‡</sup>PRISM, Université Versailles St Quentin,  
Versailles France  
Email : sayo@prism.uvsq.fr

**Abstract**—We have designed a tool to partition a Markov Chain and have used the censoring technique and strong stochastic comparison to obtain bounds on rewards and the first passage time. We present the main ideas of the method, the algorithms, the tool and some numerical results.

## I. INTRODUCTION

We present *CUT*, a software tool we have developed to bound some rewards on Discrete Time Markov Chains (DTMCs) and which is based on strong stochastic bounds and Markov chain censoring. The tool is a set of C programs to build a finite DTMC which can provide stochastic upper bounds for some steady-state and transient rewards and the first passage time as well. The original model may be a finite or infinite ergodic Markov chain. Previous algorithms to numerically compute stochastic bounds have used the lumpability approach (see [2] for the algorithms and [3] for the tool associated) and only considered finite DTMC. Here we show how we can compute bounds using the Censored Markov Chain (CMC) approach and how we can deal with large and even infinite Markov Chains. We first give in the next chapter an introduction to CMCs and to the stochastic complement matrix and we present the key idea for several algorithms recently developed.

## II. CMCs AND STOCHASTIC BOUNDS

Consider a discrete time irreducible Markov chain  $\{X_n : n = 1, 2, \dots\}$  with state space  $S$ . Suppose that  $S = E \cup E^c$ ,  $E \cap E^c = \emptyset$  and the subset  $E$  is finite. Suppose that the successive visits of  $X_n$  to  $E$  take place at time epochs  $0 < n_1 < n_2 < \dots < \dots$ . Then the chain  $\{X_t^E = X_{n_t}, t = 1, 2, \dots\}$  is called the censored chain (CMC) with censoring set  $E$  [8]. Let  $Q$  denote the transition matrix of chain  $X_n$ . Consider the partition of  $S$  to obtain a block description of  $Q$  :

$$Q = \begin{pmatrix} Q_E & Q_{EE^c} \\ Q_{E^cE} & Q_{E^cE^c} \end{pmatrix} \begin{matrix} E \\ E^c \end{matrix} \quad (1)$$

The CMC only watches the chain when it is in  $E$ . The matrix of the CMC is (Th. 2 in [8]) :

$$S_E = Q_E + Q_{EE^c} \left( \sum_{i=0}^{\infty} (Q_{E^cE^c})^i \right) Q_{E^cE} \quad (2)$$

Assume that  $(Q_{E^cE^c})$  does not contain any recurrent class, the fundamental matrix is  $\sum_{i=0}^{\infty} (Q_{E^cE^c})^i = (I - Q_{E^cE^c})^{-1}$ . CMCs

have also been called restricted or watched Markov chains. When the chain is ergodic, there are strong relations with the theory of stochastic complement [6]. Note that it is not necessary here that the chain is ergodic and we can study the absorbing time. In many problems  $Q$  can be large or infinite and therefore it is difficult to compute  $(I - Q_{E^cE^c})^{-1}$  to finally get  $S_E$ . Deriving bounds from  $Q_E$  and some information on the other blocks without computing  $S_E$  is therefore an interesting alternative approach. In *CUT*, we have implemented several algorithms to compute bounds for CMCs : (1) Truffet's algorithm [7]; (2) many algorithms based on graphs and paths ; (3) DPY algorithm that some of us have already presented [1] and which is based on blocks  $Q_E$  and  $Q_{E^cE}$ . In the following we restrict ourselves due to the space limitation to the first two types of algorithms.

### A. Bounds

Consider two probability distributions  $p$  and  $q$ , we say that  $p$  is smaller than  $q$  in the strong stochastic sense ( $p \leq_{st} q$ ) iff  $\sum_{j=k}^n p_j \leq \sum_{j=k}^n q_j$  for  $k = 1, 2, \dots, n$ .

It is known that monotonicity [4] and comparability of the transition probability matrices yield sufficient conditions for the stochastic comparison of Markov chains and their steady-state distributions. Vincent's algorithm [4] is the simplest solution to obtain a monotone upper bounding matrix of a stochastic matrix. To build a monotone upper bound of  $Q_E$  (which is only substochastic), Truffet's method consists in the following 2 steps : first add the slack probability in the last column of  $Q_E$  to make it stochastic and then apply Vincent's algorithm to obtain a monotone upper bound  $T_E$  to  $S_E$ . Let  $T(M)$  be the stochastic matrix obtained when we apply Truffet's method on substochastic matrix  $M$ . The methods used in *CUT* are justified by the following theorem whose proof is given in [5] :

*Theorem 1:* Let  $L_E$  be an element-wise lower bound to  $S_E$ ,  $L_E \leq S_E$ . Then  $S_E \leq_{st} T(L_E)$  and for any substochastic matrix  $L_E \leq M_E \leq S_E$  we have  $S_E \leq_{st} T(M_E) \leq_{st} T(L_E)$ .

Clearly  $Q_E$  is an element-wise lower bound of  $S_E$  and the theorem generalizes Truffet's method. It also states that the more accurate the element-wise lower bound of  $S_E$ , the more accurate the stochastic upper bound of  $S_E$ . To find a better lower bound than  $Q_E$ , we must consider again the definition of the transition matrix for a CMC (Eq. 2). The fundamental

matrix clearly has a sample-path structure which can be used to obtain more accurate bounds.

*Remark 1:*  $(\sum_{i=0}^{\infty} (Q_{E^c})^i)[j, k]$  is the sum of all probability of paths entering in  $E^c$  from  $j$  and leaving it after an arbitrary number of visits inside  $E^c$  from  $k$ .

We only need to add some paths instead of generating all of them because we need element-wise lower bounds of the fundamental matrix. This is the main idea of the approach based on paths. We have adapted several well-known graph algorithms to find some paths and compute their probability. The first passage time bound is also justified by this path structure of the fundamental matrix.

*Remark 2:* Assume that we want to compute the first passage time distribution of state  $j$  in  $E$  when the initial state  $k$  is in  $E$ . In the censored chain, all the paths going through  $E^c$  appear with smaller lengths. Thus the passage time in the censored chain provides a stochastic lower bound of the real passage time.

### B. Algorithms and the design of CUT

The algorithms must find some paths which are summed up in the fundamental matrix thus there is a trade-off between their complexity and the accuracy of the stochastic complement matrix that they provide. We have developed several algorithms and data structures to deal with paths exploration. The aim is to deal with chains which are so large that the transition matrix does not fit in memory. The first step is to obtain the states and transitions of the chain from some specifications. Within the tool the transitions are described by evolution equations of states with events. We proceed by a Breadth-First Search from a chosen initial state to generate the set  $E$  of states. Each transition is associated to an event which is described by a probability that may be state-dependent and by the transitions it triggers for each of the states. The states are described by a multidimensional vector, therefore they are included into a Cartesian product. The user has to modify 4 functions written in C to specify the Cartesian product including the states, the initial state to perform the visit, the probability of a transition, and the evolution equation which describes all of the transitions. He must also provide some parameters (typically the size of the components and the maximal number of states in a path). Finally one must compile the tool.

We always assume that matrix  $Q_E$  fits in memory with sparse format. But we also have several algorithms and data-structures to deal with set  $E^c$  and the three other blocks when they fit or not in memory. We do not present here the details of the algorithms because of the lack of space but we just give some highlights (the theory and the algorithms are presented in [5]).

The first step is to remove the single loops because they do not help to find the first path from  $E$  to  $E$  going through  $E^c$ . The loops will be added at the end of the algorithm to generate a set of paths and increase the lower bound of the probability. The two basic techniques are Breadth-First Search and Shortest Path algorithms. Breadth-First Search algorithm

generates all the paths of length smaller than  $d$  while Shortest Path algorithm gives the path with the higher probability when the cost of link  $(a, b)$  is  $|\log(Q(a, b))|$ . We add a constraint on the length to avoid paths with a very large number of states (i.e.  $d$ ). Remember that we compute the shortest path according to a cost function but not the path with the shortest hop number. We compute the probability of the paths we have found and add it to the corresponding elements of  $Q_E$  to obtain an element-wise lower bound  $L_E$  to  $S_E$ . Note that we do not generate all the paths and that the path lengths are upper bounded by  $d$  so we can deal with an infinite Markov chain.

### C. An example

We present a rather abstract model to give some results and time measurements. We consider a set of  $N$  resources : they can be operational or faulty. We have two types of faults (hard/soft). The fault arrivals follow independent Poisson processes with rate  $\lambda_h = 0.0001$  and  $\lambda_s = 0.5$ . The distribution of times to fix a fault are exponential with rate  $\mu_h = 0.02$  and  $\mu_s = 1$  except when all the resources are faulty. In that case, the repairman can speed up the fixing and with rate  $\mu = 1$  all the resources are repaired. Note that the considered chain is not NCD because of the transitions with rate  $\mu$ . We present in Table 1, the probability  $p$  to have  $N$  resources operational, the upper bound on this probability and the time  $T$  (in second) to compute them. Numerical examples are obtained by gathering in set  $E$  the states with 0 hard error. We can see that computation times are drastically reduced using bounding approach. It also provides results when exact analysis fails ( $N = 10000$ ).

model size		Exact		Bound	
$N$	space size	$T$	$p$	$T$	$p$
100	5151	1.57	3.61e-6	0.17	4.12e-6
500	125751	168.47	7.51e-7	0.91	8.71e-7
1000	501501	603.14	4e-7	2.31	4.80e-7
10000	50015001	-	-	123.01	2.89e-7

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