Stochastic bounds applied to the end to end QoS in communication systems

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***Abstract*—**[1]
End to end QoS of communication systems is essential for users but their performance evaluation is a complex issue. The abstraction of such systems are usually given by multidimensional Markov processes whose analysis is very difficult and even intractable, if there is no specific solution form.

In this study, we propose an algorithm in order to automatically derive aggregated Markov processes providing upper and lower bounds on performance measures. We applied the algorithm to the analysis of an open tandem queueing network with rejection in order to derive performance measure bounds. Parametric aggregation schemes have been proposed in order to compute bounds on loss probabilities and end to end mean delays. Therefore a tradeoff between the accuracy of the bound and the size of considered Markov chains is possible.

Keywords: Markov processes, Stochastic comparisons, Tandem queueing networks, performance measures bounds.

I. Introduction

QoS is still a topic that attains a lot of attention in both wired and wireless worlds. It is evident that the future networking environment will be strongly characterized by the heterogeneity of networks, especially regarding the network access part, although having IP as the common denominator. In this paper, we propose to evaluate the performance of the whole network from the source to the destination node in order to guarantee to every user an end to end QoS.

Usually, systems are represented by multidimensional processes with very large state spaces. As a result, quantitative analysis is difficult if there is no specific solution form (product form solutions, ...). Since, exact performance measures can only be obtained using numerical methods [15] with small sizes, it is important to develop new powerful mathematical tools. In this paper, we propose to use a mathematical method based on stochastic comparisons of Markov processes. The key idea of this method is the following: given a large size Markov process, we propose to bound it by a smaller new Markov process which provides bounds on performance measures. In [1], we apply this method on mobile networks in order to obtain dropping handover bounds. In [4], we use it to compute loss rates packets in an optical switch, and in [2] for the loss rates packets in an IP switch. [11] presents in details this method and apply it to evaluate cell loss rates in an ATM switch. These different studies lead to define the main steps of the generation of an aggregated bounding Markov process on multidimensional state spaces. In the case of totally ordered state spaces, the lumpability and the stochastic ordering have been combined to derive bounding Markov chains [6], [14].

In [3], we have defined an algorithm generating aggregated Markov processes, but only providing upper bounds on performance measures as loss probabilities. In the present paper, the proposed algorithm is more general : the aggregated Markov process provides upper or lower bound on performance measures. Moreover, not only loss probabilities are computed, but end to end mean delays are also estimated. The relevance of this algorithm is that it can be applied for general multidimensional processes, endowed with only a preorder (so not necessarily a total order), on the state space. We have proved using the stochastic comparison methods that proposed aggregated Markov processes provide performance measures bounds.

As an application of our algorithm, we evaluate the performances of a system represented by a series of network nodes (switches or routers), where only one flow of packets transits. The performance study of this system is performed in order to verify that end to end Quality of Service (QoS) constraints are maintained. This system can be represented as an open tandem queueing network with rejection. This tandem queues system does not have a product-form solution. One way of analyzing such queueing system is to solve numerically for the stationary probability vector of the underlying Markov process. Meanwhile, memory complexity limits this approach to small queueing networks. Most of the studies about tandem queues are approximative methods based on system decomposition. For instance, tandem queues with blocking have been analyzed in [10] using approximation algorithms for any number of queues.

This paper is organized as follows. In the next section, we present the algorithm generating aggregated bounding Markov processes, and in section III, we prove that the algorithm constructs aggregated Markov processes providing bounds on

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performance measures. In section IV, we apply the algorithm to a tandem queues system in order to evaluate the loss probabilities and the end to end mean delays. Numerical results show that the methodology is really interesting. Main results are discussed in section V, and comments about further research items are given. Finally, we resume in an appendix the stochastic ordering theory used in this paper.

II. An algorithm for the state space reduction

The stochastic comparison is a mathematical tool which allows to compute bounds on transient distributions and the stationary distribution of a Markov process. In fact, if the underlying Markov process does not have a specific solution form like a product-form or matrix-geometric solutions, etc. the computation of stationary probability distributions becomes difficult or intractable for large state spaces. By means of the stochastic comparison method, it is possible to overcome this problem by reducing the size of the state space of the underlying Markov process.

A. The proposed approach

We focus on performance measures computed as an increasing reward function on the stationary distribution. In fact we can consider transient reward functions with stochastic comparison approach in the same way, but we give here only the stationary rewards for the sake of brevity.

Let $\{X(t),\ t \geq 0\}$ be a Markov process defined on a multidimensional and preordered (not necessarily a totally ordered) state space E, with an infinitesimal generator Q. We suppose that the process is irreducible so the stationary distribution Π exists. We denote by R a performance measure computed on Π as follows

$$R = \sum_{x \in E} \Pi(x) f(x) \tag{1}$$

where f is an increasing reward function on distribution Π according to the preorder defined on E. If there is no specific solution for distribution Π, then it is very difficult to obtain as the state space E is very large. We propose an algorithm which builds a new Markov process on a subset of E, by gathering some states and by mapping them into one state. This algorithm can be applied only if two conditions are verified. First, we need to define on the state space E a preorder $\preceq$ compatible with R (R is written as an increasing reward function f on Π according to $\preceq$). Secondly, $\{X(t),\ t \geq 0\}$ must be $\preceq_{st}$ monotone (see Definition 5 in the appendix). The main steps of the algorithm are the definition of the aggregated state spaces with the mapping functions, and the construction of the infinitesimal generator matrices of aggregated processes.

B. Aggregated bounding Markov processes

We present the algorithm which generates the aggregated Markov process $\{X^s(t), t \geq 0\}$ (resp. $\{X^l(t), t \geq 0\}$) representing the upper bound (resp. the lower bound) defined on the state space S^s (resp. S^l) with the infinitesimal generator Q^s (resp. Q^l). First, we explain how to derive the mapping function $Ag^s : E \rightarrow S^s$ (resp. $Ag^l : E \rightarrow S^l$) for the upper bound (resp. the lower bound). Some states are not aggregated which means that they are exactly represented, and others are put together in order to be mapped into one state.

• If a state x_i of E is not aggregated with other states, then it is mapped into the same state x_i. So $Ag^s(x_i) = x_i$ (resp. $Ag^l(x_i) = x_i$) for the upper bound (resp. lower bound). And x_i is called a "simple" state of S^s (resp. S^l).

• If the states of the set of states $\{x_1, \ldots, x_i, \ldots, x_n\}$ of E where $x_1 \preceq x_2 \ldots \preceq x_n$ are put together, then for the upper bound (resp. the lower bound), $Ag^s(x_1) = \ldots = Ag^s(x_i) = \ldots = Ag^s(x_n) = x_n$ and x_n is called a "macro" state of S^s, (resp. $Ag^l(x_1) = \ldots = Ag^l(x_i) = \ldots = Ag^l(x_n) = x_1$ and x_1 is called a "macro" state of S^l) .

Note that the mapping function must be defined as an increasing function, in order to have the monotonicity property verified (see section III.A).

We introduce M_{Ag^s}, the matrix representation of the mapping function Ag^s, described for Theorem 2 of the Appendix. The infinitesimal generator Q^s is defined from Q and M_{Ag^s} as follows:

$$\forall x \in S^s,\ Q^s[x, *] = Q[x, *] M_{Ag^s}$$

where $Q[x, *]$ represents the row in matrix Q corresponding to state x. And equivalently, the infinitesimal generator Q^l is computed from Q and M_{Ag^l} as follows:

$$\forall x \in S^l,\ Q^l[x, *] = Q[x, *] M_{Ag^l}$$

As we can see, the main advantage of this algorithm is to generate automatically an aggregated Markov process providing performance measure bounds. Obviously, the bounds are more accurate if Q^s and Q^l are defined close to Q. However, the definition of the mapping functions Ag^s and Ag^l, which means the choice of the states to aggregate is not simple, and is fixed after trying different aggregation schemes in order to see their impact on the quality of the bounds.

Next, we prove using the stochastic comparisons of Markov processes that our algorithm really provides aggregated bounds (upper or lower).

III. Algorithm proofs and results

Using the stochastic ordering theory presented in the Appendix, we prove that aggregated Markov processes represent bounds for the Markov process $\{X(t), t \geq 0\}$. So we have to verify that : $\{Ag^s(X(t)), t \geq 0\} \preceq_{st} \{X^s(t), t \geq 0\}$ and $\{X^l(t), t \geq 0\} \preceq_{st} \{Ag^l(X(t)), t \geq 0\}$.

We give only the proof for the lower bound, as the first one has been presented in [3]. In order to apply Theorem 2 of the Appendix, we have to prove the condition 2) which means the monotonicity of one of the processes by mapping functions.

A. The monotonicity condition

We need to define the following proposition for a Markov process $\{X(t),\ t \geq 0\}$ defined on E.

Proposition 1: If the following conditions 1 and 2 are satisfied:

1) $\{X(t),\ t \geq 0\}$ is $\preceq_{st}$-monotone
2) $f : E \to S$ is an increasing function

then $\{f(X(t)),\ t \geq\}$ is also $\preceq_{st}$-monotone

This proposition can be easily proved using the coupling of processes [7], [8]. In our case, $\{X(t),\ t \geq 0\}$ is $\preceq_{st}$-monotone, and Ag^l is an increasing function, then using Proposition 1 we deduce that $\{Ag^l(X(t)), t \geq 0\}$ is also $\preceq_{st}$-monotone. So condition 2) of Theorem 2 of the Appendix is verified.

B. Infinitesimal generator comparisons

It follows from Theorem 2 of the Appendix that we have to verify condition 3):

$$\forall x \in S^l,\ y \in E\ \mid\ x = Ag^l(y),\ \ Q^l[x,*] \preceq_{st} Q[y,*]M_{Ag^l}$$

We have two cases for a state $x \in S^l$:

• if x is a simple state, then x is mapped to itself : $x = Ag^l(x)$. In this case, it follows from the definition of Q^l that $Q^l[x,*] = Q[x,*]M_{Ag^l}$, thus for a simple state x, we have $Q^l[x,*] \preceq_{st} Q[x,*]M_{Ag^l}$.

• if x is a macro state, then $\exists x_1, \ldots x_n \in E$ such that $Ag^l(x_1) = \ldots = Ag^l(x_n) = x$. As x represents the lower state, if $x_n \succeq \ldots, \succeq x_1$, then we have $x = x_1$. In this case, Q^l is defined as $Q^l[x,*] = Q[x_1,*]M_{Ag^l}$.

Since $\{Ag^l(X(t)), t \geq 0\}$ is $\preceq_{st}$-monotone, we have $Q[x_1,*]M_{Ag^l} \preceq_{st} Q[x_2,*]M_{Ag^l} \preceq_{st} \ldots \preceq_{st} Q[x_n,*]M_{Ag^l}$ [13], [7]. Therefore $\forall 1 \leq i \leq n,\ Q^l[x,*] \preceq_{st} Q[x_i,*]M_{Ag^l}$, and for a macro state $x \in S^l$, we have :

$$Q^l[x,*] \preceq_{st} Q[y,*]M_{Ag^l},\ \forall y \in E\ \mid\ Ag^l(y) = x$$

As the inequality is true for all states $x \in S^l$, it follows from Theorem 2 of the Appendix that if the condition 1) is verified, we have:

$$\{X^l(t), t \geq 0\} \preceq_{st} \{Ag^l(X(t)), t \geq 0\}$$

The upper bound follows similarly Theorem 2 of the Appendix :

$$\{Ag^s(X(t)), t \geq 0\} \preceq_{st} \{X^s(t), t \geq 0\}$$

C. Performance measures comparison

The stochastic comparison of stochastic processes generates the stochastic comparison of transient and stationary probability distributions :

$$\Pi M_{Ag^s} \preceq_{st} \Pi^s \ \text{ and } \ \Pi^l \preceq_{st} \Pi M_{Ag^l}$$

For all performance measures $R = \sum_{x \in E} \Pi(x) f(x)$, where f is an increasing reward function on E, we have (as Ag^s is a mapping function to an upper state):

$$R = \sum_{x \in E} \Pi(x) f(x) \leq \sum_{x \in S^s} \Pi M_{Ag^s}(x) f(x)$$

and from the stochastic ordering relation we have for the upper bound:

$$\sum_{x \in S^s} \Pi M_{Ag^s}(x) f(x) \leq R^s = \sum_{x \in S^s} \Pi^s(x) f(x)$$

It is similar for the lower bound, so we can deduce that for all performance measures R written as an increasing reward function f on the stationary distribution Π, we can compute an upper (resp. lower) bound R^s (resp. R^l) on Π^s (resp. Π^l), such that : $R^l \leq R \leq R^s$.

IV. Application and numerical results

We propose to apply the proposed algorithm to the performance evaluation of an open tandem queueing network.

A. Open tandem queueing network

The system understudy represents a path in a network defined as a series of network nodes (switches, routers) where transits only one flow of packets. We suppose that the leftmost node has the index 1, and indexes increase in the path until node n. This system can be represented by n finite capacity queues in tandem. Queues are numbered from 1 to n starting from the leftmost queue.

External arrivals occur only in queue 1 , and after this flow transits in queues $2, \ldots, n$ if there is enough place in each queue. We suppose that arrivals are Poisson process in queue 1 with rate λ, and in each queue i, the service time is Exponential with rate μ_i, and the capacity is finite denoted by B_i. After a service in queue i, the packet transits to the next queue $i+1$ if there is enough place, otherwise the packet is lost.

This system is represented by a Markov process $\{X(t),\ t \geq 0\}$ on $E = \{0, \ldots, B_1\} \times \ldots \times \{0, \ldots, B_i\} \times \ldots \times \{0, \ldots, B_n\}$. Each state $x \in E$ is represented by a vector $x = (x_1, \ldots, x_i, \ldots, x_n)$, where x_i is the number of packets waiting in queue i. We suppose that the stationary distribution denoted Π exists.

The goal of this performance study is to compute the loss probabilities P_i of any queue i, and the end to end mean delay D. P_i is given by :

$$P_i = \sum_{x \in E | x_i = B_i} \Pi(x) \qquad (2)$$

For the end to end mean delay, we have the following assumptions. For each queue i we denote by: λ_i the effective arrival rate, D_i the mean delay, and by N_i the mean number of packets. The effective arrival rates are computed as follows: $\lambda_1 = \lambda(1 - P_1)$, and $\lambda_i = \lambda_{i-1}(1 - P_i)$ for $1 \leq i \leq n$. The mean delay D_i for node i is computed through the Little formula: $D_i = \frac{N_i}{\lambda_i}$, where $N_i = \sum_{x \in E} x_i \Pi(x)$. The end to end end mean delay D is obtained by summing the mean delays D_i through the path: $D = \sum_{i=1}^{n} D_i$.

The resolution of $\{X(t),\ t \geq 0\}$ in order to compute distribution Π is very difficult: there is no product-form, and the number of states increases exponentially with the number of components. We apply the algorithm to construct aggregated bounding Markov processes in order to derive the performance measure bounds. The following two conditions must be verified to apply the algorithm.

B. Algorithm conditions

The first one is the definition on the state space E of an order compatible with the performance measures. We propose the component-wise partial order:

$$\forall x,\ y \in E\ x \preceq y \Leftrightarrow x_1 \leq y_1, \ldots, x_n \leq y_n$$

We choose this preorder because it allows to make comparisons on each queue, and it is compatible with the loss probabilities P_i, and the mean number N_i in each queue i which is used for the computation of the end to end mean delays. Both can be written as increasing reward function f according to the order $\preceq$ defined on E.

From expression of P_i (equation (2)), for a state $x \in E$, the reward function f is: $f(x) = 1$ if $x_i = B_i$, and $= 0$ otherwise, thus f is an increasing reward function according to the order $\preceq$ defined on E.

The mean delay D_i is computed from the mean number N_i of packets in queue i. In this case, f is also an increasing function : it equals x_i for the state x where the ith component equals x_i, and 0 otherwise. The second condition is the monotonicity of the process then we have to prove that $\{X(t), t \geq 0\}$ is $\preceq_{st}$-monotone.

Proposition 2: The considered tandem queue with rejection is $\preceq_{st}$-monotone.

This proposition is proved in [3].

C. Performance measure bounds

We have verified the two conditions of the algorithm, then we can apply it in order to compute performance measure bounds. We define two aggregated Markov processes: the upper bound $\{X^s(t), t \geq 0\}$ on the state space $S^s \subset E$, with infinitesimal generator Q^s, and the lower bound $\{X^l(t), t \geq 0\}$ on the state space $S^l \subset E$, with infinitesimal generator Q^l. Aggregation schemes associated to the mapping functions are specified precisely in section IV.D.

State spaces S^s and S^l are generated from E, and infinitesimal generator Q^s and Q^l are computed from Q. So we obtain the aggregated Markov processes : $\{X^s(t), t \geq 0\}$ and $\{X^l(t), t \geq 0\}$ such that

$$\{Ag^s(X(t)), t \geq 0\} \preceq_{st} \{X^s(t), t \geq 0\}$$

and

$$\{X^l(t), t \geq 0\} \preceq_{st} \{Ag^l(X(t)), t \geq 0\}$$

For the performance measures, we have the following proposition :

Proposition 3: We have :

$$\forall 1 \leq i \leq n,\ P_i^l \leq P_i \leq P_i^s, \text{ and } \sum_{i=1}^{n} D_i^l \leq D \leq \sum_{i=1}^{n} D_i^s$$

Proof: As the loss probabilities P_i and the mean packet number N_i in each queue i are written as increasing reward functions on the stationary distributions, then from section III.C, we have the following inequalities :

$$\forall 1 \leq i \leq n,\ P_i^l \leq P_i \leq P_i^s \text{ and } N_i^l \leq N_i \leq N_i^s \quad (3)$$

The comparison of the loss probabilities is derived directly from stochastic comparison of the processes. For the end to end mean delays, we have to prove the inequalities. We denote by $\lambda_1^s = \lambda(1 - P_1^s)$, and for $2 \leq i \leq n$ $\lambda_i^s = \lambda_{i-1}^s(1 - P_i^s)$. Similarly, $\lambda_1^l = \lambda(1 - P_1^l)$, and for $2 \leq i \leq n$ $\lambda_i^l = \lambda_{i-1}^l(1 - P_i^l)$. Therefore, it follows from inequalities on the loss probabilities (Eq. 3): $\lambda_i^s \leq \lambda_i \leq \forall I$. From the Little formula, we have for any queue $i : D_i^s = \frac{N_i^s}{\lambda_i^s}$. Since $N_i \leq N_i^s$, and $\lambda_i \geq \lambda_i^s$, we have $D_i \leq D_i^s$.

Similarly, for the lower bound, as $N_i \geq N_i^l$ and $\lambda_i \leq \lambda_i^l$ we have : $D_i \geq D_i^l$. So we can deduce an upper bound and a lower bound for the end to end delay $D = \sum_{i=1}^{n} D_i$:

$$\sum_{i=1}^{n} D_i^l \leq D \leq \sum_{i=1}^{n} D_i^s$$

■

The relevance of this result is that we have solved the problem of obtaining the loss probabilities and the end to end mean delays by the computation of upper and lower bounds. In the next section, we give some numerical results of the loss probability bounds by considering different aggregation schemes.

D. Numerical Results

In this analysis, we are interested in the packet loss probabilities of the last queue and the end to end mean delays. We propose the following aggregation scheme in order to compute upper and lower bounds on the considered performance measures. The scheme is based on a parameter Δ (Delta) which indicates the absolute difference between the number of packets in queues i and j. The states for which the difference between the number of packets in queues i and j is greater than Δ are aggregated to *upper* states for the upper bounds and to *lower* states for the lower bounds. For lower states, it is the same principle except that, if the difference between the number of packets in queues i and j is greater than Δ, then the number of packets in queue i becomes equal to the number of packets in queue j by adding Δ. The aggregated Markov chains are directly generated by taking into account this modified dynamics and they are irreducible. Obviously, the accuracy will be better for larger values of Δ and we have the exact process if $\Delta = B$. This aggregation scheme is interesting since it lets to find a tradeoff between the accuracy of bounds and the numerical complexity.

We give numerical results for a model with four buffers in tandem. We suppose that the service rate μ_i is $100Mb/s$ in each queue, the packet size is 512 bytes and we vary the input bit rate λ from 50 Mb/s to 90 Mb/s. We give results for the Packet Loss Probabilities (PLP) of the last queue and for the end to end mean delays (MD). As the exact values can be obtained only for limited state space sizes, we could not take large queueing systems. First, we propose to compare the exact values with the upper and lower bounds for system with same capacity of buffer equal to $B_i = B = 20$ for $1 \leq i \leq 4$. The size of the exact Markov process is 194481 and we take different values of Δ.

In figure 1, we give upper bounds for $\Delta = 15$, lower bounds for $\Delta = 15$ and exact values for packet loss probabilities.

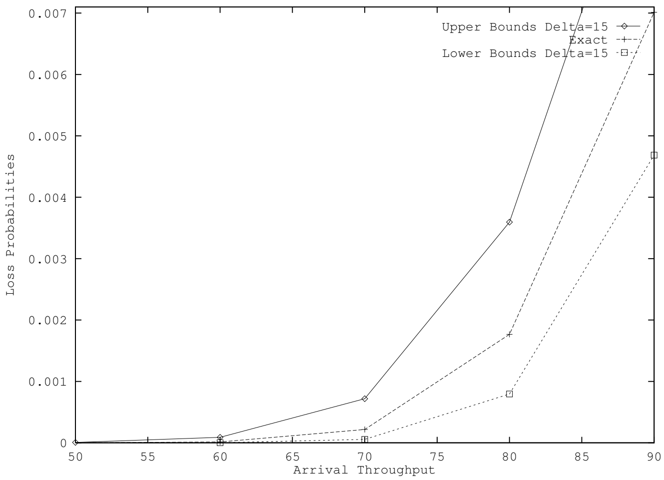

Fig. 1. PLP upper bounds and PLP lower bounds and PLP Exact values

In figure 2, we give upper bounds, lower bounds and exact values for the end to end mean delays. Two values of $Delta$ are tested : $\Delta = 10$ and $\Delta = 15$. For $\Delta = 10$, the size of the aggregated Markov process is 158071 and for $\Delta = 15$, the size is 191751.

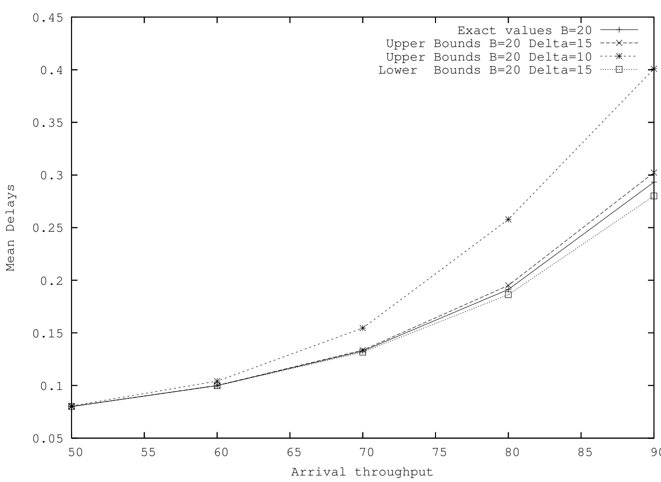

Fig. 2. MD upper bounds, lower bounds and exact values

In the both figures 1 and 2, we give exact values in order to show the quality of the bounds. The aggregated scheme provides accurate bounds.

We can notice that while Δ increases, the bounds are tighter since we approach to the exact values. On the other hand, if Δ decreases, the size of the aggregated process decreases, thus numerical analysis becomes tractable. Hence, the value of Δ lets to find a tradeoff between the accuracy of the bounds and the complexity of numerical resolution.

In figure 3, we study a system with larger buffer sizes. We take take buffer sizes equal to 40, so the size of the exact Markov chain is 2825761. Performance measures are

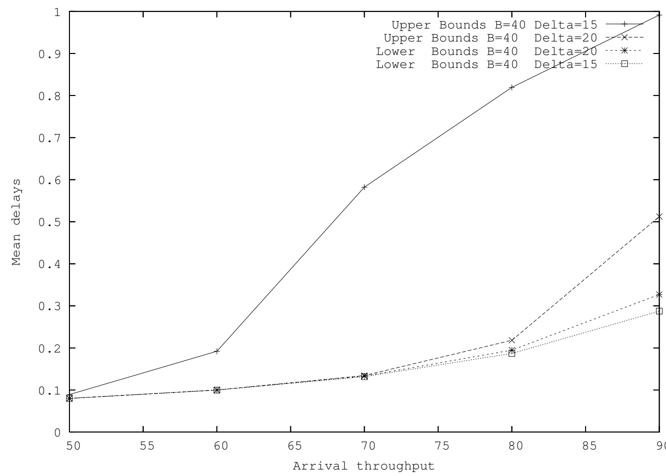

Fig. 3. MD upper bounds and lower bounds

computed for two values of Δ : $\Delta = 20$ and $\Delta = 15$. The size of aggregated chain with $\Delta = 20$ is equal to 2296141 and with $\Delta = 15$, it is equal to 1604036. In figure 3, we give upper bounds and lower bounds for $\Delta = 20$ and $\Delta = 15$. We can notice that the small difference between upper and lower bounds which let us to conclude that our aggregated scheme gives accurate bounds. Thus, this approach seems promising, we are working in the numerical analysis for large buffer sizes. Note that we have proposed in this paper a particular aggregation scheme, it is possible to consider other aggregation schemes. We are working to see the impact of these schemes on the quality of bounds.

V. Conclusion

Performance evaluation of communication systems for end to end QoS is important but complex to do. Quantitative analysis of multidimensional Markov chains may be very difficult, so we propose a general algorithm in order to reduce the state space size. The so generated aggregated Markov processes provide bounds on performance measures written as increasing reward functions on the stationary distributions. The algorithm has been applied to the analysis of an open tandem queueing network with rejection. We derive two important performance measures bounds : loss probabilities and end to end mean delays. Different aggregation schemes can be used since the constraints on the aggregated states are not restrictive. The proposed aggregation scheme lets to construct easily the underlying aggregated chains with different sizes. Thus we can have a tradeoff between the accuracy of bounds and the computation complexity. Furthermore, the proposed approach can be also applied to provide transient bounds.

Appendix

We present in this appendix some theorems and definitions about stochastic orderings used in proofs presented in this paper.

Two formalisms can be used for the definitions: increasing functions [13], [5] or increasing sets [9] .

The $\preceq_{st}$ ordering is the most known stochastic ordering, it is equivalent to the sample path ordering (see Strassen's theorem [13]). Stochastic orderings are defined only on discrete and countable state space E, where a binary relation $\preceq$ is defined at least as a preorder [13].

We consider two random variables X and Y defined respectively on E, and their probability measures given respectively by the probability vectors p and q where $p[i] = Prob(X = i),\ \forall i \in E$ (resp. $q[i] = Prob(Y = i),\ \forall i \in E$). The $\preceq_{st}$ ordering can be defined using increasing functions as follows [13]:

Definition 1: $X \preceq_{st} Y \Leftrightarrow E[(f(X))] \preceq E[(f(Y))]\ \forall f : E \rightarrow R$, $\preceq$-increasing whenever the expectations exist.

Different methods are associated to the $\preceq_{st}$ ordering: the coupling [13], [7], or the increasing set theory [9].

We present now the comparison of stochastic processes. Let $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ stochastic processes defined on E.

Definition 2: We say that $\{X(t), t \geq 0\} \preceq_{st} \{Y(t), t \geq 0\}$ if $X(t) \preceq_{st} Y(t), \forall t \geq 0$

Methods as increasing sets and coupling can also be used for Markov processes. Here we give the theorem of the coupling of the processes [7], [13]. Two processes are defined in this theorem: $\left\{\widehat{X}(t), t \geq 0\right\}$ (resp. $\left\{\widehat{Y}(t), t \geq 0\right\}$)) having the same infinitesimal generator as $\{X(t), t \geq 0\}$ (resp. $\{Y(t), t \geq 0\}$).

Theorem 1: We say that $\{X(t), t \geq 0\} \preceq_{st} \{Y(t), t \geq 0\}$ if and only if there exists the coupling $\{(\widehat{X}(t), \widehat{Y}(t)), t \geq 0\}$ such that: $\widehat{X}(t) \preceq \widehat{Y}(t),\ \forall t \geq 0$

When the processes are defined on different states spaces we can compare them on a common state space using mapping functions. Let $X(t)$ (resp. $Y(t)$) be defined on E (resp F), and g (resp. h) be a many to one mapping from E (resp. F) to S.

The stochastic comparisons of these processes by mapping functions is [5]:

Definition 3: We say that $\{g(X(t)), t \geq 0\} \preceq_{st} \{h(Y(t)), t \geq 0\}$ if $g(X(t)) \preceq_{st} h(Y(t)), \forall t \geq 0$

Mapping functions used in this paper must be $\preceq$-increasing. Next, we give the definition of an increasing function f:

Definition 4: We said that $f : E \mapsto S$ is $\preceq$-increasing if and only if: $\forall x, y \in E,\ x \preceq y \Rightarrow f(x) \preceq f(y)$

For processes defined on different states spaces, Theorem 1 can be reformulated [5]. We present in the sequel only the increasing set theory using infinitesimal generators matrices because it is the formalism developed in the algorithms presented in this paper. If we suppose that $\{X(t),\ t \geq 0\}$ (resp. $\{Y(t),\ t \geq 0\}$) is a Markov process with infinitesimal generator matrix Q_1 (resp. Q_2), then we present the theorem of the stochastic comparison of Markov processes defined on different state spaces using increasing set formalism [9], [13].

The mapping functions are represented by a matrix formalism as follows. Let M_g (resp. M_h) denote the matrix representing the mapping g (resp. h).

$$M_g[i,j]_{i \in E \text{ and } j \in S} = \begin{cases} 1 & \text{if } g(i) = j \\ 0 & \text{else} \end{cases}$$

Theorem 2: If the following conditions 1, 2 and 3 are satisfied:

1) $g(X(0)) \preceq_{st} h(Y(0))$
2) $\{g(X(t)), t \geq 0\}$ or $\{h(Y(t)), t \geq 0\}$ is $\preceq_{st}$-monotone
3) $Q_1[x,*]M_g \preceq_{st} Q_2[y,*]M_h,\ \forall x \in E, y \in F,\ g(x) = h(y)$

then we have: $\{g(X(t)), t \geq 0\} \preceq_{st} \{h(Y(t)), t \geq 0\}$

where $Q_1[x,*]$ is the row in the matrix Q_1 corresponding to the state x. And,

$$Q_1[x,*]M_g \preceq_{st} Q_2[y,*]M_h$$

is equivalent to: $\forall x \in E, y \in F \mid g(x) = h(y)$

$$\sum_{g(z) \in \Gamma} Q_1(x,z) \leq \sum_{h(z) \in \Gamma} Q_2(y,z),\ \forall \Gamma \in \Phi_{st}(S)$$

where Γ is an increasing set from $\Phi_{st}(E)$, inducing the $\preceq_{st}$ ordering [9]. Note that the stochastic comparison of Markov processes by mapping functions can be interesting for reducing the state space size of Markov processes, in order to define bounding aggregated Markov processes as we will see in this paper.

The monotonicity of the Markov process is used in condition (2) of this theorem. The monotonicity of a stochastic process is defined as an increasing in t [7].

Definition 5: We say that $\{X(t), t \geq 0\}$ is $\preceq_{st}$ -monotone if $X(s) \preceq_{st} X(t), \forall s,\ t \in \mathbb{R}^+,\ s \leq t$

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