Model Checking of Infinite State Space Markov Chains by Stochastic Bounds *

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Abstract. In this paper, we discuss how to check Probablistic Computation Tree Logic (PCTL) logic operators over infinite state Discrete Time Markov Chains (DTMC). Probabilistic model checking has been largely applied over finite state space Markov models. Recently infinite state models have been considered when underlying infinite Markov models have special structures. We propose to consider finite state models providing bounds on transient and the stationary distributions in the sense of the $\preceq_{st}$ stochastic order to check infinite state models. The operators of the PCTL logic are then checked by considering these finite bounding models.

1 Introduction

Model checking is a method to automatically check if complex performability guarantees expressed by using formal logics are satisfied or not. Stochastic model checking is a recent extension of traditional model-checking techniques for the integrated analysis of both qualitative and quantitative system properties. Model checking for different classes of stochastic processes and specification logics have been developed [4, 12, 8] and have been also implemented in different model checkers [16, 13]. However in almost all works, the state space size is considered to be finite. To perform model checking by numerical analysis we need to compute transient-state or steady-state distribution of the underlying Markov chain [5]. The numerical methods exist only for finite state models, however for special structured chains like QBD (Quasi Birth Death) models despite the infinite state spaces efficient numerical algorithms called matrix-geometric solutions exist. In [19], Continuous Stochastic Logic (CSL) over Continuous Time Markov Chains (CTMC) model checking has been extended to infinite space QBD models, and in [20] models with product-form solutions have been considered.

In this paper we propose to consider model checking of general infinite Markov chains with the stochastic comparison techniques. These techniques have

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been largely applied in different areas of applied probability as well as in reliability, performance evaluation, dependability applications [17]. Intuitively speaking, this method consists in computing bounding distributions rather than the exact distributions by analysing "simpler" bounding chains.

Bounding methods can be applied in model checking context since one needs to verify if some thresholds are satisfied or not without computing the exact values. In [2], the bounds on state reachability probabilities of Markov decision processes are computed by abstraction of the underlying model defined on smaller state spaces. If the verification of the considered property can not be concluded, the abstract model is refined until a verdict to the property can be deduced from the computations. The stochastic comparison techniques have been applied in [18, 7] to overcome state space explosion problem in the model checking context. In [18], PRCTL [3] state formulas are considered by using stochastic bounding techniques. In [7], a method to simplify the checking of CSL operators by means of class $\mathcal{C}$ bounding Markov chains having closed-form solutions for transient and the steady-state distribution is given.

Our approach in this work is based on the truncation of the underlying infinite chains which are intractable and the computation of finite stochastic matrices providing, via stochastic comparison, bounds on the relevant probability quantities for the model checking. The idea of truncation is very natural and seems necessary to be able to deal numerically with general infinite DTMCs. It has been proposed to compute approximations of stationary distributions of infinite Markov chains [21, 14]. However approximations are not useful for model checking. Moreover we need to compute bounds also on transient distributions in order to check path formulas and transient operators.

In the model checking context we must sum the probabilities of a set of states from a distribution of the underlying model. This set of states depends on the considered formula and the distribution is the steady-state distribution for the stationary operator while we must consider a transient distribution for a path formula. The stochastic comparison has the advantage of providing bounds on the steady-state as well as transient distributions. Moreover the $\preceq_{st}$ stochastic order considered in this work allows to deduce bounds on the partial sums of distributions. In this paper, since we must establish the stochastic comparison of distributions, one having finite size and the other having the infinite size, we apply the stochastic comparison of the images of these distributions on a common space. We present this method and discuss its application to check different formulas depending also on the comparison operator $\leq$ or $\geq$.

The remaining of the paper is organised as follows: In section 2, we give a brief introduction on the stochastic comparison approach and the Probabilistic real time Computation Tree Logic (PCTL) for Discrete Time Markov Chains (DTMC)s. Section 3 is devoted to the bounding of infinite DTMCs by finite DTMCs. We explain in section 4, how these bounds can be used to check PCTL operators over infinite DTMCs.

2 Preliminaries

2.1 Stochastic Comparison

Stochastic comparison is an useful tool to compare random variables and stochastic processes when studying stochastic systems. First, we give the basic definitions and theorems letting to compare random variables and DTMCs defined on the *same state space* with respect to the usual stochastic order $\preceq_{st}$. Secondly we present an interesting extension, called fg-comparison that we apply in this work. This extension introduced by Doisy [9] allows to compare random variables and DTMCs which are not defined on the same state space by means of *state functions* f and g. For further informations on the stochastic comparison we refer to Stoyan's book [17] as the main reference in the domain and to the works [9], [10] and [22] for the fg-comparison.

Comparison of DTMCs on the same state space

Definition 1 *Let X and Y be two random variables (r.v) taking values on a totally ordered space E, and $\mathcal{F}_{st}$ the class of all increasing real functions on E.*
$X \preceq_{st} Y \Longleftrightarrow Ef(X) \leq Ef(Y), \quad \forall f \in \mathcal{F}_{st}$ *whenever the expectations exist.*

Property 1. For two r.v X and Y taking values on a totally ordered space E
$X \preceq_{st} Y \quad \Longleftrightarrow \quad Prob(X > a) \leq Prob(Y > a), \forall a \in E$

In the case of finite state space $\{0, 1, \ldots, N\}$, the $\preceq_{st}$-comparison of random variables can be characterised through the following probability inequalities.

Property 2. Let X and Y be two r.v taking values on $E = \{0, 1, \ldots, N\}$, and $p = [p_0, \ldots, p_N]$, $q = [q_0, \ldots, q_N]$ be probability distributions of X and Y.

$$X \preceq_{st} Y \Leftrightarrow \sum_{k=i}^{N} p_k \leq \sum_{k=i}^{N} q_k \quad for \quad i = N, N-1, \cdots, 0. \tag{1}$$

$$X \preceq_{st} Y \Leftrightarrow \sum_{k=0}^{j} p_k \geq \sum_{k=0}^{j} q_k \quad for \quad j = 0, 1, \cdots, N. \tag{2}$$

Let us remark that in the sequel we interchangeably use the notations $X \preceq_{st} Y$ and $p \preceq_{st} q$. We apply the following definition to compare Markov chains.

Definition 2 *Let $\{X(n)\}$ (resp. $\{Y(n)\}$) be a DTMC. We say $\{X(n)\} \preceq_{st} \{Y(n)\}$, if $X(n) \preceq_{st} Y(n), \ \forall n$.*

In the case of time-homogeneous DTMC chains, the monotonicity and the comparability of transition matrices yield sufficient conditions to compare stochastically the underlying chains [17, p.186].

Theorem 1 *Let* $\mathbf{P}$ *(resp.* $\mathbf{Q}$*) be the transition matrix of the time-homogeneous Markov chain* $\{X(n)\}$ *(resp.* $\{Y(n)\}$*). The comparison of Markov chains is established (*$\{X(n)\} \preceq_{st} \{Y(n)\}$*), if the following conditions are satisfied :*

i- $X(0) \preceq_{st} Y(0)$,

ii- at least one of the probability transition matrices is monotone, that is, either $\mathbf{P}$ *or* $\mathbf{Q}$ *is* $\preceq_{st}$ *monotone :*
$\forall i, j$ *such that* $i \leq j$*, either* $\mathbf{P}[i,*] \preceq_{st} \mathbf{P}[j,*]$ *or* $\mathbf{Q}[i,*] \preceq_{st} \mathbf{Q}[j,*]$

iii- the transition matrices are comparable in the sense of the $\preceq_{st}$ *order :*

$$\mathbf{P} \preceq_{st} \mathbf{Q} \Longleftrightarrow \mathbf{P}[i,*] \preceq_{st} \mathbf{Q}[i,*], \quad \forall i \in E$$

where $\mathbf{P}[i,*]$ *denotes the* i*th row of matrix* $\mathbf{P}$.

In the following property, we give the comparison of Discrete-Time Markov chains (DTMCs) in terms of distributions for the sake of readability.

Property 3. Let $\{X(n)\}$ (resp. $\{Y(n)\}$) be a DTMC, $\mathbf{\Pi_X^n}$ (resp. $\mathbf{\Pi_Y^n}$) its transient distribution at time n, and $\mathbf{\Pi_X}$ (resp. $\mathbf{\Pi_Y}$) its steady-state distribution (if it exists). If $\{X(n)\} \preceq_{st} \{Y(n)\}$ then $\mathbf{\Pi_X^n} \preceq_{st} \mathbf{\Pi_Y^n}$, $\forall n$ and $\mathbf{\Pi_X} \preceq_{st} \mathbf{\Pi_Y}$.

fg-Comparison of DTMCs We now define the fg-comparison between two probability measures p and q which are not defined on the same state space. Let p (resp. q) be defined on the state space E (resp. F); f (resp. g) be a surjective function from E (resp. F) into a state space G; $\{X(n)\}$ (resp. $\{Y(n)\}$) be a time-homogeneous DTMC defined on the discrete space E (resp. F) with transition matrix P (resp. Q). The fg-comparison between p and q is defined as follows:

Definition 3

$$p \preceq_{st}^{fg} q \quad \Longleftrightarrow \quad \tilde{p} \preceq_{st} \tilde{q}$$

where $\tilde{p} = fp$ *is the image measure of* p *by* f $(\forall i \in G, \tilde{p}_i = \sum_{j \in E, f(j)=i} p_j)$.

The following example illustrates this type of comparison: $E = \{1,2,3,4\}$, $F = \{1,2,3\}$ and $G = F$. The function $f : E \rightarrow F$ is defined as $f(1) = 1$, $f(2) = f(3) = 2$ and $f(4) = 4$ and g is the identity function. Hence, $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = (p_1, p_2{+}p_3, p_4)$ and $(p_1, p_2, p_3, p_4) \preceq_{st}^{fg} (q_1, q_2, q_3) \Longleftrightarrow (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \preceq_{st} (q_1, q_2, q_3) \Longleftrightarrow \sum_{k=i}^{3} \tilde{p}_k \leq \sum_{k=i}^{3} q_k, i = 3, 2, 1$. For instance, $(0.4, 0.2, 0.3, 0.1) \preceq_{st}^{fg} (0.35, 0.45, 0.2)$.

Definition 4 *The DTMC* $\{X(n)\}$ *is said to be less than the DTMC* $\{Y(n)\}$ *with respect to the order* $\preceq_{st}^{fg}$*, if* $X(n) \preceq_{st}^{fg} Y(n)$*,* $\forall n$.

Definition 5 $\mathbf{P}$ *is* $\preceq_{st}^{f}$*-monotone if and only if,* $\mathbf{P}[x,*] \preceq_{st}^{f} \mathbf{P}[y,*]$*,* $\forall x, y \in E$*, such that* $f(x) \leq f(y)$

Definition 6 $\mathbf{P} \preceq_{st}^{fg} \mathbf{Q}$ *if and only if,* $\forall x \in E, \forall y \in F$ *such that* $f(x) = g(y)$*,* $\mathbf{P}[x,*] \preceq_{st}^{fg} \mathbf{Q}[y,*]$

Theorem 2 $\{X(n)\} \preceq_{st}^{fg} \{Y(n)\}$ *if the following conditions are satisfied :*
$X(0) \preceq_{st}^{fg} Y(0)$, $\mathbf{P} \preceq_{st}^{fg} \mathbf{Q}$ and $\mathbf{P}$ is $\preceq_{st}^{f}$-monotone or $\mathbf{Q}$ is $\preceq_{st}^{g}$-monotone.

We have the $\preceq_{st}^{fg}$ comparison between transient distributions and the state-state distribution, if the underlying chains are comparable in this sense:

Property 4. Let $\{X(n)\}$ (resp. $\{Y(n)\}$) be a DTMC, $\mathbf{\Pi^n_X}$ (resp. $\mathbf{\Pi^n_Y}$) its transient distribution at time n, and $\mathbf{\Pi_X}$ (resp. $\mathbf{\Pi_Y}$) its steady-state distribution (if it exists). If $\{X(n)\} \preceq_{st}^{fg} \{Y(n)\}$ then $\mathbf{\Pi^n_X} \preceq_{st}^{fg} \mathbf{\Pi^n_Y}$, $\forall n$ and $\mathbf{\Pi_X} \preceq_{st}^{fg} \mathbf{\Pi_Y}$.

2.2 Model checking DTMC

In this subsection, we briefly present the logic called *Probabilistic real time Computation Tree Logic* (PCTL) [12] which allows to express formulas over discrete time Markov chains.

DTMC and notations Throughout this paper, the considered DTMCs may be finite or infinite with a countable state space. A labelled finite (resp. infinite) DTMC $\mathcal{M}$ is a 3-tuple $(S, \mathbf{P}, L)$ where S is a finite (resp. infinite countable) set of *states*, $\mathbf{P} : S \times S \to \mathcal{R}^+$ is the *transition matrix* and $L : S \to 2^{AP}$ is a *labelling* function which assigns to each state $s \in S$, the set $L(s)$ of atomic propositions $a \in AP$ that are valid in s, AP denotes the set of atomic propositions.

For a DTMC, there are two types of state probabilities : transient probabilities where the system is considered at time n and steady-state probabilities when the system reaches an equilibrium if it exists. In the sequel, $\mathbf{\Pi}_\alpha^\mathcal{M}(s', n)$ denotes the probability to be in state s' at time n with initial distribution α. $\mathbf{\Pi}_\alpha^\mathcal{M}(s') = \lim_{n\to\infty} \mathbf{\Pi}_\alpha^\mathcal{M}(s', n)$ is the steady-state probability to be in state s'. If $\mathcal{M}$ is ergodic, $\mathbf{\Pi}_\alpha^\mathcal{M}(s')$ exists and it is independent of the initial distribution, so we will denote it by $\mathbf{\Pi}^\mathcal{M}(s')$. For Markov chain $\mathcal{M}$ we denote by $\mathbf{\Pi}_\alpha^\mathcal{M}(n)$ the transient distribution at time n when the initial distribution is α and by $\mathbf{\Pi}^\mathcal{M}$ the steady-state distribution. For $S' \subseteq S$, we denote by $\mathbf{\Pi}_\alpha^\mathcal{M}(S', n)$ (resp. $\mathbf{\Pi}^\mathcal{M}(S')$) the transient probability to be in states of S', $\mathbf{\Pi}_\alpha^\mathcal{M}(S', n) = \sum_{s'\in S'} \mathbf{\Pi}_\alpha^\mathcal{M}(s', n)$ (the steady-state probability to be in states of S', $\mathbf{\Pi}^\mathcal{M}(S') = \sum_{s'\in S'} \mathbf{\Pi}^\mathcal{M}(s')$). In the case of an unique initial state s (i.e. $\alpha(s) = 1$ and $\alpha(s') = 0$ for $s \neq s'$), we simply write $\mathbf{\Pi}_\alpha^\mathcal{M}(n)$ by $\mathbf{\Pi}_s^\mathcal{M}(n)$.

A path through a DTMC $\mathcal{M}$ can be finite or infinite. For instance a finite path σ of length k is a sequence of states $\sigma = s_0, s_1, \cdots, s_k$ with $s_i \in S$ and $\mathbf{P}(s_i, s_{i+1}) > 0$ $\forall i$. We denote by $paths_s$ the set of all paths starting from state s and by $\sigma[i]$ the i^{th} state s_i of the path σ.

Syntax of PCTL We give here the syntax of PCTL as defined in [12] and its extension by a steady-state operator that has been proposed in [3]. Let n be an integer, p a probability and $\triangleleft$ a comparison operator $\in \{\leq, \geq\}$. In the sequel, we denote by S_ϕ or ϕ-states the set of states that satisfy ϕ and by $\models$ the satisfaction relation. The syntax of PCTL is:

$$\phi ::= true \mid a \mid \phi \wedge \phi \mid \neg\phi \mid \mathcal{P}_{\lhd p}(\phi\; \mathcal{U}^{\leq n}\phi) \mid \mathcal{S}_{\lhd p}(\phi)$$

In this paper, for the sake of simplicity, we do not consider the next state operator and the other Boolean connectives (false, $\vee$, $\Rightarrow$) that are derived in the usual way. Let us present the semantics of these formulas as defined in [12]:

$s \models true$	for all $s \in S$
$s \models a$	iff $a \in L(s)$
$s \models \neg\phi$	iff $s \not\models \phi$
$s \models \phi_1 \wedge \phi_2$	iff $s \models \phi_1 \wedge s \models \phi_2$
$s \models \mathcal{P}_{\lhd p}(\phi_1\; \mathcal{U}^{\leq n}\phi_2)$	iff $Prob^{\mathcal{M}}(s, \phi_1\mathcal{U}^{\leq n}\phi_2) \lhd p$
$s \models \mathcal{S}_{\lhd p}(\phi)$	iff $\mathbf{\Pi}_s^{\mathcal{M}}(S_\phi) \lhd p$

$Prob^{\mathcal{M}}(s, \phi_1\mathcal{U}^{\leq n}\phi_2)$ denotes the probability measure of the paths σ starting in s ($\sigma \in paths_s$) satisfying $\phi_1\; \mathcal{U}^{\leq n}\phi_2$ i.e. $Prob^{\mathcal{M}}(s, \phi_1\mathcal{U}^{\leq n}\phi_2) = Prob\{\sigma \in paths_s \mid \sigma \models \phi_1\; \mathcal{U}^{\leq n}\phi_2\}$. $\mathcal{P}_{\lhd p}(\phi_1\; \mathcal{U}^{I}\phi_2)$ asserts that the probability measure of paths satisfying $\phi_1\; \mathcal{U}^{\leq n}\phi_2$ meets the bound given by $\lhd p$. The path formula $\phi_1\; \mathcal{U}^{\leq n}\phi_2$ asserts that ϕ_2 will be satisfied within n time units and that all preceding states satisfy ϕ_1, i.e.

$$\sigma \models \phi_1\; \mathcal{U}^{\leq n}\phi_2 \text{ iff } \exists i \leq n \text{ such that } \sigma[i] \models \phi_2 \text{ and } \forall j < i, \sigma[j] \models \phi_1$$

$\mathcal{S}_{\lhd p}(\phi)$ asserts that the steady-state probability for ϕ-states meets the bound $\lhd p$. Similar to the steady-state operator $\mathcal{S}_{\lhd p}(\phi)$, we define a transient-state operator $\mathcal{T}_{\lhd p}^{@n}(\phi)$ such that: $s \models \mathcal{T}_{\lhd p}^{@n}(\phi)$ iff $\mathbf{\Pi}_s^{\mathcal{M}}(S_\phi, n) \lhd p$.

Checking PCTL operators In [12], a methodology has been proposed to check bounded until operator, $\mathcal{P}_{\lhd p}(\phi_1\mathcal{U}^{\leq n}\phi_2)$. Let us consider the following partition of S into three subsets:

- the success states, are labelled with ϕ_2
- the failures states, are states which are not labelled with ϕ_1 nor ϕ_2
- the inconclusive states, are states labelled with ϕ_1 but not with ϕ_2

Let $\mathcal{M}[\phi]$ be the DTMC defined from $\mathcal{M} = (S, \mathbf{P}, L)$, by making all ϕ-states (states satisfying ϕ) in $\mathcal{M}$ absorbing, i.e. $\mathcal{M}' = (S, \mathbf{P}', L)$ where $\mathbf{P}'(s, s') = \mathbf{P}(s, s'), \forall s' \in S$ if $s \not\models \phi$ and if $s \models \phi$ then $\mathbf{P}'(s, s) = 1$ and $\mathbf{P}'(s, s') = 0$. It has been shown that $Prob^{\mathcal{M}}(s, \phi_1\; \mathcal{U}^{\leq n}\phi_2)$ can be computed by means of transient distributions of DTMC $\mathcal{M}'$ which is obtained from $\mathcal{M}$ by making success states and failures states absorbing. In fact once a success state is reached before n time units, $\phi_1\mathcal{U}^{\leq n}\phi_2$ is satisfied regardless of which states will be visited in the future. On the other hand, $\phi_1\mathcal{U}^{\leq n}\phi_2$ is violated once a failure state is visited. Formally, $\mathcal{M}' = \mathcal{M}[\neg\phi_1 \wedge \neg\phi_2][\phi 2] = \mathcal{M}[\neg\phi_1 \vee \phi_2]$ and we have the equation $Prob^{\mathcal{M}}(s, \phi_1\; \mathcal{U}^{\leq n}\phi_2) = \sum_{s' \models \phi_2} \mathbf{\Pi}_{\mathbf{s}}^{\mathcal{M}'}(s', n)$ ([15]).

$\mathbf{\Pi}_{\mathbf{s}}^{\mathcal{M}'}(s', n)$ denotes the probability of reaching state s' in n steps in the DTMC $\mathcal{M}'$ when starting in s. Consequently,

$$s \models \mathcal{P}_{\lhd p}(\phi_1 \, \mathcal{U}^{\leq n} \phi_2) \quad \text{iff} \quad \mathbf{\Pi}_{\mathbf{s}}^{\mathcal{M}'}(S_{\phi_2}, n) \lhd p \tag{3}$$

To check the steady-state operator $\mathcal{S}_{\lhd p}(\phi)$ (resp. the transient-state operator $\mathcal{T}_{\lhd p}^{@n}(\phi)$) it suffices to verify that the steady state probability (resp. the transient distribution probability at time n) to be in ϕ states, meets the bound $\lhd p$:

$$s \models \mathcal{S}_{\lhd p}(\phi) \quad \text{iff} \quad \mathbf{\Pi}_{s}^{\mathcal{M}}(S_{\phi}) \lhd p \tag{4}$$

$$s \models \mathcal{T}_{\lhd p}^{@n}(\phi) \quad \text{iff} \quad \mathbf{\Pi}_{s}^{\mathcal{M}}(S_{\phi}, n) \lhd p \tag{5}$$

3 Bounding Infinite DTMCs by Finite DTMCs

In the sequel, $\{Y(n)\}$ denotes an infinite state space, time-homogeneous DTMC taking values in $\{0, 1, 2, \ldots\}$, with transition matrix $R = (r_{i,j})_{i,j \geq 0}$. We want to define a finite DTMC $\{X(n)\}$ such that $\{X(n)\} \preceq_{st}^{fg} \{Y(n)\}$ for some state functions f and g. In this section, we will give two such lower bounding finite DTMCs. The first bound is valid with a monotonicity condition on the transition matrix R while there is no condition for the second bound. Since a time-homogeneous DTMC is completely defined by its transition matrix and its initial distribution, the proposed bounding chains ($\{X(n)\}$) are given in terms of their transition matrices and initial distributions. Let us remark here that the bounding algorithms given in this section are inspired from bounding algorithms [1, 11], so we do not give their proofs. Moreover their complexities in the worst-case without any sparse implementation neither optimisation is quadratic. We first give the definition of partial monotonicity which is required for the first bound.

Definition 7 *A transition matrix P is said partially $\preceq_{st}$-monotone from level K, if* $\quad P[i, *] \preceq_{st} P[j, *] \quad \forall i, j \geq K$ *such that* $i < j$

3.1 First bound

We first construct a finite state-space transition matrix by truncating the underlying infinite state transition matrix, R at state N and by augmenting the probabilities of column N to make the truncated matrix stochastic. By doing so, we do not remove states greater than N but they are aggregated to state N. Let $Q = (q_{i,j})_{0 \leq i,j \leq N}$ be the matrix defined by :

$$q_{i,j} = \begin{cases} r_{i,j}, & 0 \leq i \leq N, 0 \leq j \leq N-1 \\ \sum_{k \geq N} r_{i,k}, & 0 \leq i \leq N, j = N \end{cases} \tag{6}$$

In the sequel we call N *the truncation level* and Q *the stochastic truncated matrix of R at level N*. Remark that the quantity $\sum_{k \geq N} r_{i,k}$ can be easily computed since it is equal to $1 - \sum_{k=0}^{N-1} r_{i,k}$.

The second step consists in constructing $P = (p_{i,j})_{0 \le i,j \le N}$ through Algorithm 1. The input parameters are the truncated matrix Q, and the probability vector q defined from row $N{+}1$ of R as $q = (r_{N+1,0}, \ldots, r_{N+1,N-1}, \sum_{k \ge N} r_{N+1,k})$. Therefore P is a stochastic matrix which is $\preceq_{st}$-monotone, lower bound in the sense of $\preceq_{st}$ of Q ($P \preceq_{st} Q$) and the Nth row of P is less than vector q in the sense of $\preceq_{st}$ order ($P[N,*] \preceq_{st} q$). Let $\nu = (\nu_0, \nu_1, \nu_2, \ldots)$ be the initial

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Algorithm 1:

Input : stochastic matrix A; probability vector p.

Output : B such that 1)stochastic matrix, 2)$\preceq_{st}$-monotone, 3)$B \le_{st} A$, 4)$B[N,*] \preceq_{st} p$.

1 $b_{N,0} = \max(a_{N,0}, p_0)$

for $i = N-1$ **downto** 0 **do**

2     $b_{i,0} = \max(a_{i,0}, b_{i+1,0})$

end

for $j = 1$ to N **do**

3     $b_{N,j} = \max(\sum_{k=0}^{j} a_{N,k}, \sum_{k=0}^{j} p_k) - \sum_{k=0}^{j-1} b_{N,k}$

    for $i = N-1$ **downto** 0 **do**

4         $b_{i,j} = max(\sum_{k=0}^{j} a_{i,k}, \sum_{k=0}^{j} b_{i+1,k}) - \sum_{k=0}^{j-1} b_{i,k}$

    end

end

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distribution of the Markov chain $\{Y(n)\}$, i.e., the distribution of $Y(0)$. Probability vector $\mu = (\mu_0, \mu_1, \ldots, \mu_N)$ is defined on $\{0, 1, \ldots, N\}$ such that $\mu_i = \nu_i$, if $i < N$ and $\mu_N = \sum_{k \ge N} \nu_k$. Let define the state spaces $E = \{0, 1, \ldots, N\}$ and $F = \{0, 1, 2, \ldots\}$. f is the identity function on E ($f(i) = i, \forall i \in E$) and function $g : F \to E$ is defined as: $g(i) = i$ if $i < N$ and $g(i) = N$ if $i \ge N$. We now demonstrate that the truncated finite state transition matrix constructed as explained above provides a lower bound on the infinite state transition matrix.

Proposition 1 *Let $\{Y(n)\}$ be an infinite DTMC with state space $\{0, 1, 2, \ldots\}$ and a transition matrix R which is partially $\preceq_{st}$-monotone from level $N+1$. If $\{X(n)\}$ is a finite DTMC with state space $E = \{0, 1, \ldots, N\}$ defined by the initial distribution μ and the transition matrix P as given above, then $\{X(n)\} \preceq_{st}^{fg} \{Y(n)\}$.*

Proof. Let m (resp. n) be a probability measure defined on E (resp. F). Using definition 3, $m \preceq_{st}^{fg} n \iff \tilde{m} \preceq_{st} \tilde{n}$. The image measure $\tilde{m}$ of m by f is equal to m ($\tilde{m}_i = m_i, \forall i \in E$) since f is the identity function. $\tilde{n} = gn$ the image measure of n by g is defined as $\tilde{n}_i = \sum_{k \in F, g(k) = i} n_k, \forall i \in E$. Hence, $\tilde{n}_i = n_i$ if $i < N$ and $\tilde{n}_N = \sum_{k \ge N} n_k$ and

$$m \preceq_{st}^{fg} n \iff (m_0, \ldots, m_{N-1}, m_N) \preceq_{st} (n_0, \ldots, n_{N-1}, \sum_{k \ge N} n_k) \qquad (7)$$

Since $\mu = (\nu_0, \ldots, \nu_{N-1}, \sum_{k \geq N} \nu_k)$, it is obvious from the last equation that $\mu \preceq_{st}^{fg} \nu$ then $X(0) \preceq_{st}^{fg} Y(0)$. P is constructed by Algorithm 1, thus it is $\preceq_{st}$-monotone : $P[x,*] \preceq_{st} P[y,*], \forall x, y \in E$ such that $x \leq y$ (see ii- of theorem 1). f is the identity function, so $P[x,*] \preceq_{st}^{f} P[y,*]$, $\forall x, y \in E$, such that $f(x) \leq f(y)$. It follows from definition 5 that P is $\preceq_{st}^{f}$-monotone.

It remains to prove that $P \preceq_{st}^{fg} R$. From definition 6, we must show that $\forall x \in E, \forall y \in F$ such that $x = g(y)$, $P[x,*] \preceq_{st}^{fg} R[y,*]$. This is equivalent to show that $\forall i < N, P[i,*] \preceq_{st}^{fg} R[i,*]$ and $P[N,*] \preceq_{st}^{fg} R[k,*], \forall k \geq N$. By construction of P, we have $P \preceq_{st} Q$, $(\forall i \leq N, P[i,*] \preceq_{st} Q[i,*])$. By definition of matrix Q (equation 6), $\forall i \leq N, Q[i,*] = (r_{i,0}, \ldots, r_{i,N-1}, \sum_{k \geq N} r_{i,k})$. Hence, $\forall i \leq N, P[i,*] \preceq_{st} (r_{i,0}, \ldots, r_{i,N-1}, \sum_{k \geq N} r_{i,k})$ and we conclude from equation 7 that

$$\forall i \leq N, P[i,*] \preceq_{st}^{fg} R[i,*]$$

On the other hand, R is supposed to be partially $\preceq_{st}$-monotone from level $N+1$. Thus, $R[N+1,*] \preceq_{st} R[k,*], \forall k > N$. By property 1, we deduce that $\sum_{j \geq N} r_{N+1,j} \leq \sum_{j \geq N} r_{k,j}, \forall k > N$ and from property 2 we have $q = (r_{N+1,0}, \ldots, r_{N+1,N-1}, \sum_{j \geq N} r_{N+1,j}) \preceq_{st} (r_{k,0}, \ldots, r_{k,N-1}, \sum_{j \geq N} r_{k,j}), \forall k > N$. By construction of matrix P with Algorithm 1, $P[N,*] \preceq_{st} q$, ie. $P[N,*] \preceq_{st} (r_{k,0}, \ldots, r_{k,N-1}, \sum_{j \geq N} r_{k,j}), \forall k > N$. Thus $P[N,*] \preceq_{st}^{fg} R[k,*], \forall k > N$, and it follows from theorem 2 that $\{X(n)\} \preceq_{st}^{fg} \{Y(n)\}$.

3.2 Second bound

The second bound is constructed by truncating the infinite state space at state N and by augmenting the probabilities of column 0 to make the truncated matrix stochastic. Let $S = (s_{i,j})_{0 \leq i,j \leq N}$ be the truncated matrix defined as :

$$s_{i,j} = \begin{cases} r_{i,j}, & 0 \leq i \leq N, 1 \leq j \leq N \\ r_{i,0} + \sum_{k > N} r_{i,k}, & 0 \leq i \leq N, j = 0 \end{cases} \quad (8)$$

Then we construct a monotone lower bounding matrix $\underline{S}$ for S through Algorithm 2. Thus $\underline{S}$ is $\preceq_{st}$-monotone and $\underline{S} \preceq_{st} S$. Matrix $T = (t_{i,j})_{0 \leq i,j \leq N}$ is obtained from $\underline{S}$ by replacing its first row by the probability distribution $(1, 0, \ldots, 0)$. Obviously, T is a stochastic matrix which is $\preceq_{st}$-monotone, and $T \preceq_{st} S$.

Let $\nu = (\nu_0, \nu_1, \nu_2, \ldots)$ be the initial distribution of the Markov chain $\{Y(n)\}$ $(Y(0))$. The probability vector $u = (u_0, u_1, \ldots, u_N)$ is defined on $\{0, 1, \ldots, N\}$ as $u_0 = \nu_0 + \sum_{k > N} \nu_k$ and $u_i = \nu_i$ if $1 \leq i \leq N$. Let define state spaces $E = \{0, 1, \ldots, N\}$ and $F = \{0, 1, 2, \ldots\}$. f is the identity function on E $(f(i) = i, \forall i \in E)$ and $h : F \to E$ is defined as: $h(i) = i$ if $i \leq N$ and $h(i) = 0$ if $i > N$.

Proposition 2 *Let $\{Y(n)\}$ be an infinite DTMC with a transition matrix $R = (r_{i,j})_{i,j \geq 0}$. If $\{X(n)\}$ is a finite DTMC with state space $E = \{0, 1, \ldots, N\}$ defined by initial distribution u and transition matrix T as given above, then $\{X(n)\} \preceq_{st}^{fh}$ $\{Y(n)\}$.*

Algorithm 2:

Input : stochastic matrix A.
Output : B such that 1) stochastic matrix, 2) $\preceq_{st}$-monotone, 3)$B \preceq_{st} A$.

$b_{N,0} = a_{N,0}$
for $i = N-1$ **downto** 0 **do**
  $b_{i,0} = \max(a_{i,0}, b_{i+1,0})$
end
for $j = 1$ to N **do**
  $b_{N,j} = \sum_{k=0}^{j} a_{N,k} - \sum_{k=0}^{j-1} b_{N,k}$
  for $i = N-1$ **downto** 0 **do**
    $b_{i,j} = \max(\sum_{k=0}^{j} a_{i,k}, \sum_{k=0}^{j} b_{i+1,k}) - \sum_{k=0}^{j-1} b_{i,k}$
  end
end

Proof. Let m (resp. n) be a probability measure defined on E (resp. F). $m \preceq_{st}^{fh} n \iff \tilde{m} \preceq_{st} \tilde{n}$ (definition 3). $\tilde{m} = m$ and $\tilde{n} = hn$ the image measure of n by h is defined as $\tilde{n}_i = \sum_{k \in F, h(k)=i} n_k, \forall i \in E$. Hence, $\tilde{n}_i = n_i$ if $i \in \{1, \ldots, N\}$, $\tilde{n}_0 = n_0 + \sum_{k>N} n_k$ and

$$m \preceq_{st}^{fh} n \iff (m_0, m_1, \ldots, m_N) \preceq_{st} (n_0 + \sum_{k>N} n_k, n_1 \ldots, n_N) \qquad (9)$$

From this equation it is clear that $u \preceq_{st}^{fh} \nu$, i.e $X(0) \preceq_{st}^{fh} Y(0)$. Similar to the proof of proposition 1, T is $\preceq_{st}^{f}$-monotone. To prove $T \preceq_{st}^{fh} R$, we must show that $\forall x \in E, \forall y \in F$ such that $x = h(y)$, $T[x,*] \preceq_{st}^{fh} R[y,*]$. This is equivalent to show that $T[0,*] \preceq_{st}^{fh} R[0,*]$, $T[0,*] \preceq_{st}^{fh} R[i,*], \forall i > N$ and that $\forall i \in \{1, \ldots, N\}, T[i,*] \preceq_{st}^{fh} R[i,*]$.

We have $T[0,*] = (1, 0, \ldots, 0) \preceq_{st} (r_{i,0} + \sum_{k>N} r_{i,k}, r_{i,1}, \ldots, r_{i,N}), \forall i > N$ (property 2). It follows from equation 9 that $T[0,*] \preceq_{st}^{fh} R[i,*], \forall i > N$. By construction of T, we have $T \preceq_{st} S$, i.e, $\forall i \leq N, T[i,*] \preceq_{st} S[i,*]$. By definition of matrix S (equation 8), $S[i,*] = (r_{i,0} + \sum_{k>N} r_{i,k}, r_{i,1}, \ldots, r_{i,N}), \forall i \leq N$. Hence, $T[i,*] \preceq_{st} (r_{i,0} + \sum_{k>N} r_{i,k}, r_{i,1}, \ldots, r_{i,N}), \forall i \leq N$. By equation 9, $T[i,*] \preceq_{st}^{fh} R[i,*], \forall i \leq N$. Finally, it follows from theorem 2 that $\{X(n)\} \preceq_{st}^{fh} \{Y(n)\}$.

4 Checking Infinite DTMCs by Stochastic Comparison

In this section we propose to check PCTL operators over infinite discrete-time Markov chains using the stochastic lower bounds given in the previous section. Throughout this section, $\{Y(n)\}$ is the underlying infinite DTMC for which we want to check PCTL operators. Before introducing the checking procedures, we first give the following proposition for the monotonicity properties of the transition matrix when some states are permuted.

Property 5. Let N be a given integer. If the transition matrix R is $\preceq_{st}$-monotone, then the infinite transition matrix R_N obtained by permuting some states larger than N is partially $\preceq_{st}$-monotone from level $N+1$.

In fact by permuting some states of the matrix R we may loose the monotonicity property and R_N may not be monotone. However, since the permutations concern only states which are not larger than N, it is obvious that R_N is partially $\preceq_{st}$-monotone from level $N+1$.

It can be seen from equations 3,4 and 5 that to check the formula $Fr = \{\mathcal{P}_{\lhd p}(\phi_1\ \mathcal{U}^{\leq n}\phi_2),\ \mathcal{S}_{\lhd p}(\phi),\ \mathcal{T}^{@n}_{\lhd p}(\phi)\}$ we have to sum the probabilities of a set of states. We denote by S_Σ this set of states and by $P_{Fr}(S_\Sigma)$ the probability of S_Σ states for the considered formula Fr. For instance, for $Fr = \mathcal{P}_{\lhd p}(\phi_1 \mathcal{U}^{\leq n}\phi_2)$, S_Σ is the set of ϕ_2 states ($S_\Sigma = S_{\phi_2}$) and $P_{Fr}(S_\Sigma) = \mathbf{\Pi}^{\mathcal{M}'}_{\mathbf{s}}(S_{\phi_2}, n)$ (see equation 3).

In general $P_{Fr}(S_\Sigma)$ is intractable because of the infinite state space except the cases where the the underlying DTMC has some special structures. We propose to compute bounds on $P_{Fr}(S_\Sigma)$ by considering finite state DTMCs. Let $P^{low}_{Fr}(S_\Sigma) \leq P_{Fr}(S_\Sigma) \leq P^{up}_{Fr}(S_\Sigma)$. Depending on the comparison operator $\lhd$, we can deduce if the underlying formula is checked or not through these bounds :

- $\lhd = \leq$: If $P^{up}_{Fr}(S_\Sigma) \leq p$, we can deduce that the underlying formula is checked.
- $\lhd = \geq$: If $P^{low}_{Fr}(S_\Sigma) \geq p$, we can deduce that the underlying formula is checked.
- In the other cases, it is not possible to decide the satisfaction or not and we must refine the bounds by increasing the truncation level.

We apply the stochastic bounding approach to derive the bounding values $P^{up}_{Fr}(S_\Sigma)$, $P^{low}_{Fr}(S_\Sigma)$. Indeed the $\preceq_{st}$ comparison of probability vectors allows to establish the inequalities between the partial sum of probabilities (see equations 1, 2). Thus the set S_Σ must be reordered at the beginning or at the end of the state space depending whether we want to obtain an upper or a lower bound on the partial sum.

The finite state space lower bounding DTMC are constructed by truncating the underlying infinite state space DTMC as explained in section 3. However it is not possible to take into account all of the possible cases depending on the comparison operator and the finiteness or not of the set S_ϕ. In the following subsections, we discuss how we can check by this method the operators $\mathcal{P}_{\lhd p}(\phi_1\ \mathcal{U}^{\leq n}\phi_2)$, $\mathcal{S}_{\lhd p}(\phi)$, $\mathcal{T}^{@n}_{\lhd p}(\phi)$. Let us remark here that the same notations as in section 3 are used in the sequel.

4.1 Checking $\mathcal{P}_{\lhd p}(\phi_1\ \mathcal{U}^{\leq n}\phi_2)$

To check $\mathcal{P}_{\lhd p}(\phi_1\ \mathcal{U}^{\leq n}\phi_2)$, we first make the success states and the failures states absorbing to obtain the DTMC $\mathcal{M}'$ (see subsection 2.2). In addition, as it has been proposed in [12] we aggregate the success states and the failures states into two representative macro-states $s_{success}$ and $s_{failures}$ which are absorbing. Let $\mathcal{M}''$ be the DTMC obtained after these transformations. From equation 3 we deduce that

$$s \models \mathcal{P}_{\triangleleft p}(\phi_1 \ \mathcal{U}^{\leq n} \phi_2) \quad \text{iff} \quad \mathbf{\Pi}_{\mathbf{s}}^{\mathcal{M}''}(s_{success}, n) \triangleleft p \tag{10}$$

We distinguish in the following the cases when the set of inconclusive states is finite and infinite. The case when the set of inconclusive states is finite is interesting. Indeed we can compute $\mathbf{\Pi}_{\mathbf{s}}^{\mathcal{M}''}(s_{success}, n)$ exactly since the DTMC $\mathcal{M}''$ is finite. If the set of inconclusive states is infinite, we first choose an integer N sufficiently large to take into account the macro-states $s_{success}$ and $s_{failures}$ and a maximum of inconclusive states and also the initial state. In the following $\{Y(n), n \geq 0\}$ is the infinite DTMC corresponding to $\mathcal{M}''$ with transition matrix R. We distinguish the cases of the comparison operators $\triangleleft = \geq$ and $\triangleleft = \leq$.

- $\underline{\triangleleft = \geq}$: We permute the macro state $s_{success}$ with state N. Let R_N be the matrix obtained after permutation. Let S be the truncated matrix of R_N at level N (equation 8). By proposition 2, $\{X(n)\} \preceq_{st}^{fh} \{Y(n)\}$ and $\pi_u^X(n) \preceq_{st}^{fh} \pi_\nu^Y(n)$. By definition of functions f and h and equation 9, this is equivalent to $(\pi_0^X(n), \pi_1^X(n), \ldots, \pi_N^X(n)) \preceq_{st} (\pi_0^Y(n) + \sum_{k>N} \pi_k^Y(n), \pi_1^Y(n), \ldots, \pi_N^Y(n))$. The success states are aggregated in state N, so we deduce from equation 1 that $\pi_N^X(n) \leq \pi_N^Y(n) = \mathbf{\Pi}_{\mathbf{s}}^{\mathcal{M}''}(s_{success}, n)$. Thus if $\pi_N^X(n) \geq p$ then $\mathbf{\Pi}_{\mathbf{s}}^{\mathcal{M}''}(s_{success}, n) \geq p$ and $\mathcal{P}_{\geq p}(\phi_1 \ \mathcal{U}^{\leq n} \phi_2)$ is satisfied, otherwise we cannot conclude.
- $\underline{\triangleleft = \leq}$: We permute the macro state $s_{success}$ with state 1. Let R_N be the matrix obtained after permutation. Using the second bound we have $\{X(n)\} \preceq_{st}^{fh} \{Y(n)\}$ and $\pi_u^X(n) \preceq_{st}^{fh} \pi_\nu^Y(n)$. By definition of functions f and h and equation 9, this is equivalent to $(\pi_0^X(n), \pi_1^X(n), \ldots, \pi_N^X(n)) \preceq_{st} (\pi_0^Y(n) + \sum_{k>N} \pi_k^Y(n), \pi_1^Y(n), \ldots, \pi_N^Y(n))$. We deduce from equation 2 that $\pi_0^X(n) + \pi_1^X(n) \geq \pi_0^Y(n) + \sum_{k>N} \pi_k^Y(n) + \pi_1^Y(n) \geq \pi_1^Y(n)$. Recall that with the considered permutation the success states are aggregated in state 1, so $\pi_0^X(n) + \pi_1^X(n) \geq \mathbf{\Pi}_{\mathbf{s}}^{\mathcal{M}''}(s_{success}, n)$. Thus if $\sum_{k=0}^{1} \pi_k^X(n) \leq p$ then $\mathbf{\Pi}_{\mathbf{s}}^{\mathcal{M}''}(s_{success}, n) \leq p$ and $\mathcal{P}_{\leq p}(\phi_1 \ \mathcal{U}^{\leq n} \phi_2)$ is satisfied, otherwise we cannot conclude.

4.2 Checking $\mathcal{T}_{\triangleleft p}^{@n}(\phi)$

We distinguish the cases of the comparison operators $\triangleleft = \geq$ and $\triangleleft = \leq$ and the cases when the set of ϕ-states, S_ϕ is finite and infinite:

- $\underline{\triangleleft = \geq \text{ and } S_\phi \text{ is finite}}$: In this case we first choose a truncation level, N sufficiently large to take into account all ϕ-states. Let m be the cardinal of S_ϕ. We first permute the ϕ-states with states from $N-m+1$ to N. Let R_N be the matrix obtained after permutation and S be the stochastic truncated matrix of R_N at level N (equation 8). We consider the same notations as in subsection 3.2 for T, u, f, h and the finite DTMC $\{X(n), n \geq 0\}$. By proposition 2, $\{X(n)\} \preceq_{st}^{fh} \{Y(n)\}$ and $\pi_u^X(n) \preceq_{st}^{fh} \pi_\nu^Y(n)$. By definition of functions f and h and equation 9, this is equivalent to $(\pi_0^X(n), \pi_1^X(n), \ldots, \pi_N^X(n)) \preceq_{st} (\pi_0^Y(n) + \sum_{k>N} \pi_k^Y(n), \pi_1^Y(n), \ldots, \pi_N^Y(n))$. The ϕ-states are states from $N-m+1$ to N, so we deduce from equation 1 that $\mathbf{\Pi}_u^X(S_\phi, n) = \sum_{k=N-m+1}^{N} \pi_k^X(n) \leq \sum_{k=N-m+1}^{N} \pi_k^Y(n) = \mathbf{\Pi}_\nu^Y(S_\phi, n)$. Thus if $\mathbf{\Pi}_u^X(S_\phi, n) \geq p$ then $\mathbf{\Pi}_\nu^Y(S_\phi, n) \geq p$ and $\mathcal{T}_{\geq p}^{@n}(\phi)$ is satisfied, otherwise we cannot conclude.

- $\lhd = \geq$ and S_ϕ is infinite : In this case we are obliged to truncate also the ϕ-states. We choose the truncation level, N sufficiently large in order to take more ϕ-states. Let S'_ϕ be the subset of ϕ-states of cardinal m which are less than N ($S'_\phi = \{s,\ 0 \leq s \leq N, s \models \phi\}$). In the same way as the previous case, we can show that $\mathbf{\Pi}_u^X(S'_\phi, n) = \sum_{k=N-m+1}^{N} \pi_k^X(n) \leq \sum_{k=N-m+1}^{N} \pi_k^Y(n) = \mathbf{\Pi}_\nu^Y(S'_\phi, n) \leq \mathbf{\Pi}_\nu^Y(S_\phi, n)$. Thus if $\mathbf{\Pi}_u^X(S'_\phi, n) \geq p$ then $\mathbf{\Pi}_\nu^Y(S_\phi, n) \geq p$ and $\mathcal{T}_{\geq p}^{@n}(\phi)$ is satisfied, otherwise we cannot conclude.
- $\lhd = \leq$ and S_ϕ is finite : In this case we first choose an integer N sufficiently large so that all states satisfying ϕ are less than N. Let m be the cardinal of S_ϕ. We permute the ϕ-states with states from 1 to m. Let R_N be the matrix obtained after permutation. Let S be the stochastic truncated matrix of R_N at level N (equation 8). By proposition 2, $\{X(n)\} \preceq_{st}^{fh} \{Y(n)\}$ and $\pi_u^X(n) \preceq_{st}^{fh} \pi_\nu^Y(n)$. By definition of functions f and h and equation 9, this is equivalent to $(\pi_0^X(n), \pi_1^X(n), \ldots, \pi_N^X(n)) \preceq_{st} (\pi_0^Y(n) + \sum_{k>N} \pi_k^Y(n), \pi_1^Y(n), \ldots, \pi_N^Y(n))$. We deduce from equation 2 that $\sum_{k=0}^{m} \pi_k^X(n) \geq \pi_0^Y(n) + \sum_{k>N} \pi_k^Y(n) + \sum_{k=1}^{m} \pi_k^Y(n) \geq \sum_{k=1}^{m} \pi_k^Y(n)$. Recall that with the considered permutation the ϕ-states are states from 1 to m, so $\sum_{k=0}^{m} \pi_k^X(n) \geq \mathbf{\Pi}_\nu^Y(S_\phi, n)$. Thus if $\sum_{k=0}^{m} \pi_k^X(n) \leq p$ then $\mathbf{\Pi}_\nu^Y(S_\phi, n) \leq p$ and $\mathcal{T}_{\leq p}^{@n}(\phi)$ is satisfied, otherwise we cannot conclude. In the case when transition matrix R is in addition $\preceq_{st}$-monotone, we can construct an upper bound for $\mathbf{\Pi}_\nu^Y(S_\phi, n)$ using the first bound exactly in the same way as for the steady state operator with the comparison operator $\lhd = \leq$ (see section 4.3). Thus we can use two bounds for checking $\mathcal{T}_{\leq p}^{@n}(\phi)$ in this case.
- $\lhd = \leq$ and S_ϕ is infinite: If $S_{\neg\phi}$ is finite, we can check $\mathcal{T}_{\leq p}^{@n}(\phi)$. Let m be the cardinal of $S_{\neg\phi}$. We permute the m states not-satisfying ϕ with states from 1 to m. We take the level of truncation N equal to m. Let R_N be the matrix obtained after permutation. Let S be the stochastic truncated matrix of R_N at level N (equation 8). By proposition 2, $\{X(n)\} \preceq_{st}^{fh} \{Y(n)\}$ and $\pi_u^X(n) \preceq_{st}^{fh} \pi_\nu^Y(n)$. By definition of functions f and h and equation 9, this is equivalent to $(\pi_0^X(n), \pi_1^X(n), \ldots, \pi_N^X(n)) \preceq_{st} (\pi_0^Y(n) + \sum_{k>N} \pi_k^Y(n), \pi_1^Y(n), \ldots, \pi_N^Y(n))$. We deduce that $\pi_0^X(n) \geq \pi_0^Y(n) + \sum_{k>N} \pi_k^Y(n)$. Remark that with the considered permutation the set S_ϕ is constituted by state 0 and all states greater than N. Thus $\pi_0^X(n) \geq \mathbf{\Pi}_\nu^Y(S_\phi, n)$ and if $\pi_0^X(n) \leq p$ then $\mathbf{\Pi}_\nu^Y(S_\phi, n) \leq p$ and $\mathcal{T}_{\leq p}^{@n}(\phi)$ is satisfied, otherwise we cannot conclude.

4.3 Checking $\mathcal{S}_{\triangleleft p}(\phi)$

By this methodology, this operator can be checked only when R is stochastically monotone, the set of ϕ-states is finite and the comparison operator $\triangleleft = \leq$. Let us remark that the second bound is not interesting for the steady-state case, since we make the first state absorbing.

We first choose an integer N sufficiently large so that all states satisfying ϕ are less than N. Let m be the cardinal of S_ϕ. We put the m ϕ-states at the beginning of the state space, i.e, we permute the ϕ-states with states from 0 to $m-1$. Let R_N be the matrix obtained after permutation. This matrix is partially $\preceq_{st}$-monotone from level $N+1$ (proposition 5). Let Q be the stochastic truncated matrix of R_N at level N (equation 6). By proposition 1, $\{X(n)\} \preceq_{st}^{fg} \{Y(n)\}$ and $\pi^X \preceq_{st}^{fg} \pi^Y$ By definition of functions f and g and equation 7, this is equivalent to $(\pi_0^X, \ldots, \pi_{N-1}^X, \pi_N^X) \preceq_{st} (\pi_0^Y, \ldots, \pi_{N-1}^Y, \sum_{k \geq N} \pi_k^Y)$. When the first m ϕ-states are in the beginning, we deduce from equation 2 that $\sum_{k=0}^{m-1} \pi_k^X \geq \sum_{k=0}^{m-1} \pi_k^Y = \mathbf{\Pi}^Y(S_\phi)$. Thus if $\sum_{k=0}^{m-1} \pi_k^X \leq p$ then $\mathbf{\Pi}^Y(S_\phi) \leq p$ and $\mathcal{S}_{\leq p}(\phi)$ is satisfied, otherwise we cannot conclude.

5 Conclusions

In this paper we propose an approach based on the stochastic comparison to check PCTL operators over infinite DTMCs. We present two algorithms to construct finite bounding matrices from the original Markov chain and show that these bounding matrices provide useful inequalities for checking PCTL formulas. The first proposed bound requires a monotonicity condition on the underlying matrix but it is used only for the steady state operator.

The stochastic comparison approach has been applied in general to simplify the analyse of complex systems. In this work, this approach is may be the unique alternative to deal with general infinite DTMCs which are intractable. However, the proposed method can be also used to simplify model checking of large finite DTMCs by considering smaller ones. The stochastic comparison approach has interesting potentials to perform model checking for infinite state models. We envisage to consider some case studies to illustrate the feasibility of the proposed approach and to study the tightness of the bounds. Also, we will investigate other bounding schemes based on this approach.

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