Synthesis for prefix first-order logic on data words

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Motivation

- We want an **unbounded number of agents**...
  - processes
  - computers in a network
  - drones

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- ...to satisfy some **specification**

System and **Environment**, playing actions (a and b for System, c and d for Environment) in turn on shared or proper agents :

(1,a)(8,b)(7,d)(4,c)(6,a)(6,c)(7,a)(6,d)(2,b)(7,d)(7,a)

Executions : finite or infinite **data words**

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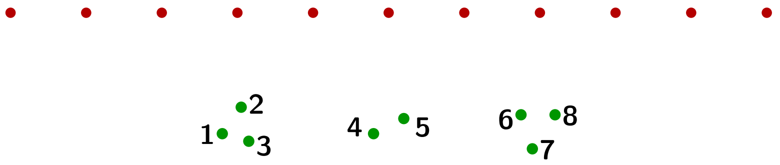
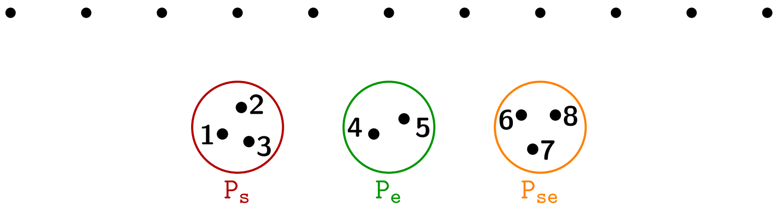


- One element for each position
- One element for each agent

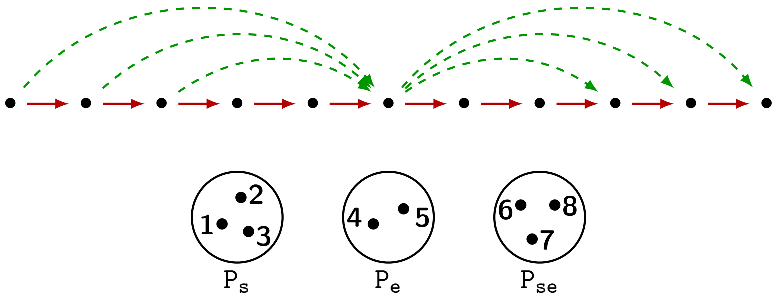
(1,a)(8,b)(7,d)(4,c)(6,a)(6,c)(7,a)(6,d)(2,b)(7,d)(7,a)



- Three unary relations P_s, P_e and P_{se} to denote ownership of the agents

Executions : finite or infinite **data words**

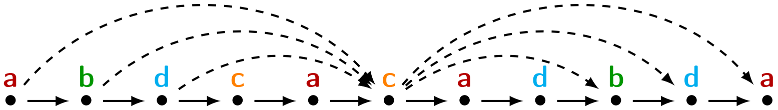
`(1,a)(8,b)(7,d)(4,c)(6,a)(6,c)(7,a)(6,d)(2,b)(7,d)(7,a)`



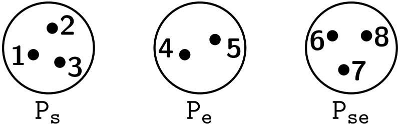
- A binary relation $+1$ between successive positions
- A binary relation $<$ for its transitive closure

Executions : finite or infinite **data words**

(1,a)(8,b)(7,d)(4,c)(6,a)(6,c)(7,a)(6,d)(2,b)(7,d)(7,a)



$A_s = \{a, b\}$
$A_e = \{c, d\}$



- A unary relation for each action

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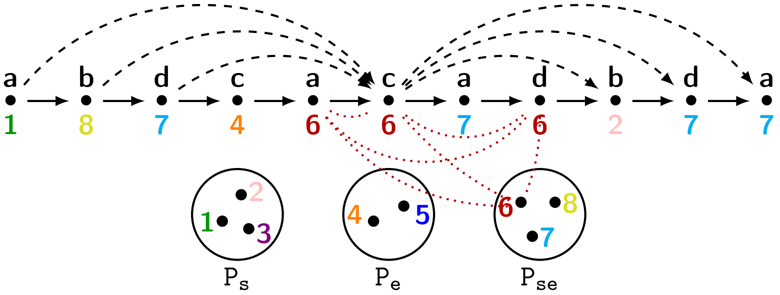

- An equivalence relation ∼ with a class for each agent

Specification language

Fragment of first-order logic , with a subset of the binary predicates

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- $\mathsf{FO}^2[\sim, <, +1]$

Specification language

$\overbrace{\text{Fragment of first-order logic}}^{\text{two variables}}$, with a subset of the binary predicates

- $\mathrm{FO}^2[\sim, <, +1]$

Specification language

Fragment of first-order logic , with a $\overbrace{\text{subset of the binary predicates}}^{\text{all predicates}}$

- $\mathsf{FO}^2[\sim, <, +1]$

Specification language

Fragment of first-order logic , with a subset of the binary predicates

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> *Every agent requesting a resource eventually gets it :*

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> $\forall x,\ \mathtt{req(x)} \rightarrow\ \exists y,\ y \sim x\ \wedge\ y > x\ \wedge\ \mathtt{gets(y)}$

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Specification language

no restriction

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Specification language

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  Every System agent requests at most twice a resource :

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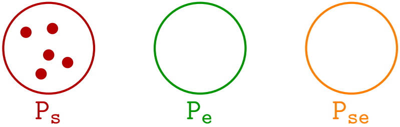
- $\mathrm{FO}[\sim]$

> *Every System agent requests at most twice a resource :*
>
> $\forall x, \mathtt{P_s}(x) \rightarrow \left[\forall y_1, y_2, y_3,\ \bigwedge_i \left(x \sim y_i \wedge \mathtt{req}(y_i)\right) \rightarrow \bigvee_{i \neq j} y_i = y_j\right]$

Agent control

We consider three configurations :

1. All the agents belong to System



Agent control

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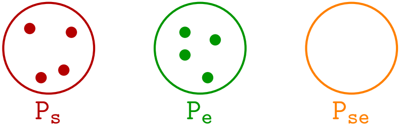
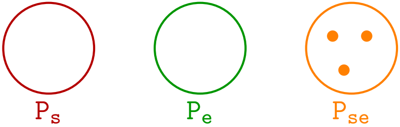
1. All the agents belong to System
2. There is no shared agent



Agent control

We consider three configurations :

1. All the agents belong to System
2. There is no shared agent
3. All the agents are shared by System and Environment



Synthesis problem

Parameters :

- a logic (specification language) $\mathcal{L}$
- a configuration for agent control (System only, partitioned or shared)

Synthesis problem

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- a logic (specification language) $\mathcal{L}$
- a configuration for agent control (System only, partitioned or shared)

Synthesis problem for $\mathcal{L}$ for this configuration :

Input : a formula $\varphi \in \mathcal{L}$

Question : does there exist a distribution of agents, complying with the configuration, such that System has a winning strategy for φ ?

Decidability boundary

Logic\Agents	System only [a]	Partitioned	Shared
$FO^2[\sim]$	decidable[1]	decidable[3]	undecidable[3]
$FO[\sim]$	decidable[2]	decidable[3]	undecidable[2]
$FO^2[\sim, <]$	decidable[1]	undecidable[3]	undecidable[3]
$FO^2[\sim, +1]$	decidable[1]	undecidable[3]	undecidable[3]
$FO^2[\sim, <, +1]$	decidable[1]	undecidable[3]	undecidable[2]

1 : [Bojańczyk et al. '06]
2 : [Bérard et al. '20]
3 : [Grange, Lehaut '23]

a. this amounts to the satisfiability problem

Decidability boundary

Logic\Agents	Partitioned
$FO^2[\sim]$	decidable
$FO[\sim]$	decidable
$FO^2[\sim, <]$	undecidable
$FO^2[\sim, +1]$	undecidable
$FO^2[\sim, <, +1]$	undecidable

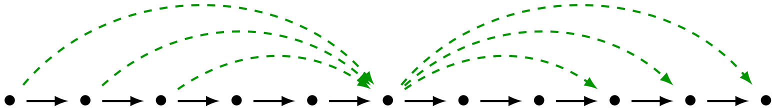
Decidability boundary

Logic\Agents	Partitioned
$\mathsf{FO}^2[\sim]$	decidable
$\mathsf{FO}[\sim]$	decidable
$\mathsf{FO}^{\mathsf{pref}}[\lesssim]$	?
$\mathsf{FO}^2[\sim,<]$	undecidable
$\mathsf{FO}^2[\sim,+1]$	undecidable
$\mathsf{FO}^2[\sim,<,+1]$	undecidable

Prefix first-order logic on *words* : FO^{pref}

Definition (FO^{pref})

As $FO[<]$ on words, where...

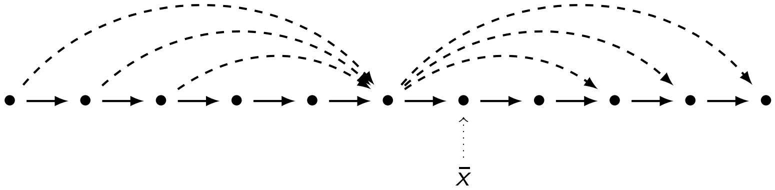


Prefix first-order logic on *words* : $\mathrm{FO}^{\mathrm{pref}}$

Definition ($\mathrm{FO}^{\mathrm{pref}}$)

As FO[<] on words, where...

- first quantifier : $\forall \bar{x}$ or $\exists \bar{x}$



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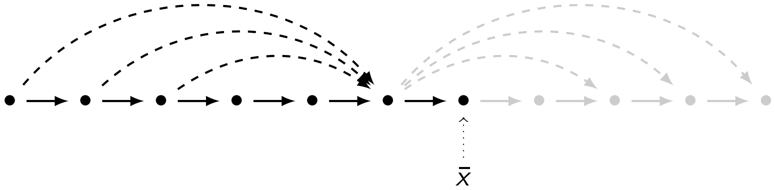

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Some factor never appears before some other factor

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Some factor never appears before some other factor

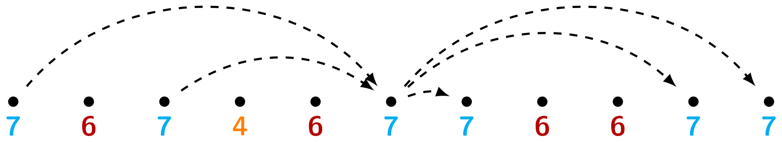
Not expressible in $\mathrm{FO}^{\mathrm{pref}}$:

There is an infinite number of a

Prefix first-order logic on *data words* : $\mathrm{FO}^{\mathrm{pref}}[\lesssim]$

Definition ($\mathrm{FO}^{\mathrm{pref}}[\lesssim]$)

$\mathrm{FO}^{\mathrm{pref}}[\lesssim]$: $\mathrm{FO}^{\mathrm{pref}}$ independently on each data class



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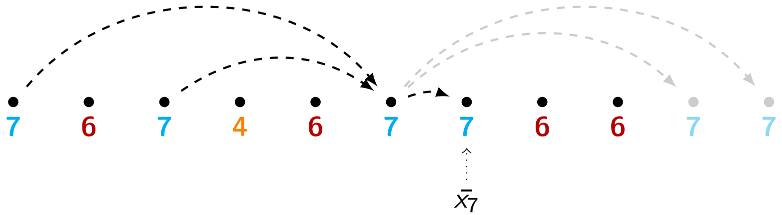

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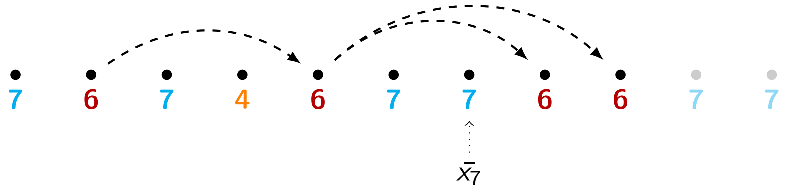

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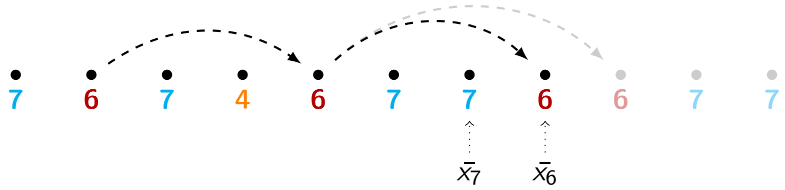

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An agent only closes a resource they opened and did not already close :

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An agent only closes a resource they opened and did not already close :

$$\forall \bar{x}, \mathrm{close}(\bar{x}) \rightarrow \big(\exists x \lesssim \bar{x}, \mathrm{open}(x) \ \wedge\ \forall y \lesssim \bar{x}, x \lesssim y \rightarrow \neg \mathrm{close}(y)\big)$$

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Not expressible in $\mathrm{FO}^{\mathrm{pref}}[\lesssim]$:

Two agents never have the same resource open simultaneously

An agent always ends up closing an open resource

Synthesis problem for $\mathsf{FO}^{\mathsf{pref}}[\lesssim]$

Input : a formula $\varphi \in \mathsf{FO}^{\mathsf{pref}}[\lesssim]$

Question : does there exist a distribution of partitioned agents such that System has a winning strategy for φ ?

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The synthesis problem for $\mathrm{FO}^{\mathrm{pref}}[\lesssim]$ *with partitioned agents is decidable*

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Sketch of proof :

1. normalize the game (strict alternation between players)

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Sketch of proof :

1. normalize the game
2. convert it to a token game
3. solve the token game (by showing it admits some kind of cutoff)

Synthesis problem for $\mathrm{FO}^{\mathrm{pref}}[\lesssim]$

Definition ($\mathrm{FO}^{\mathrm{pref}}$ type of a word w)

Set of sentences of $\mathrm{FO}^{\mathrm{pref}}$

- with as many nested quantifiers as φ
- satisfied by w

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$\mathrm{FO}^{\mathrm{pref}}$ types are stationary :

Lemma

For every infinite word w, there exists $i \in \mathbb{N}$ such that for every $j \geq i$, w and $w[1 \ldots j]$ have the same $\mathrm{FO}^{\mathrm{pref}}$ type

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Conversion to token game :

Arena : set of $\mathrm{FO}^{\mathrm{pref}}$ types (finite)

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Transitions : $\tau \to \tau'$ iff there exists a finite word w of type τ,

there exists u such that wu has type τ'

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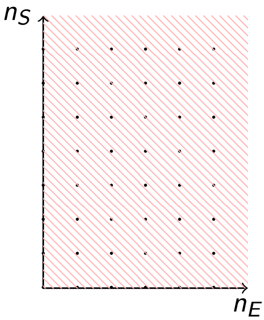
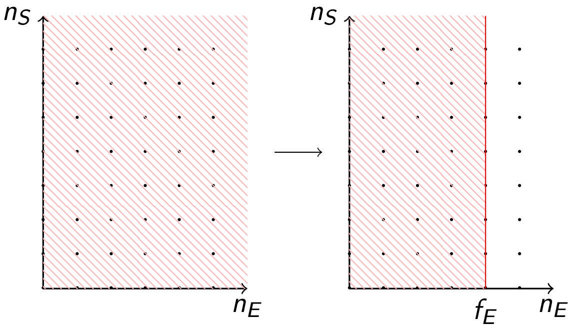
Win config : set of configurations,

with token counting up to the quantifier nesting of φ

Synthesis problem for $FO^{pref}[\lesssim]$



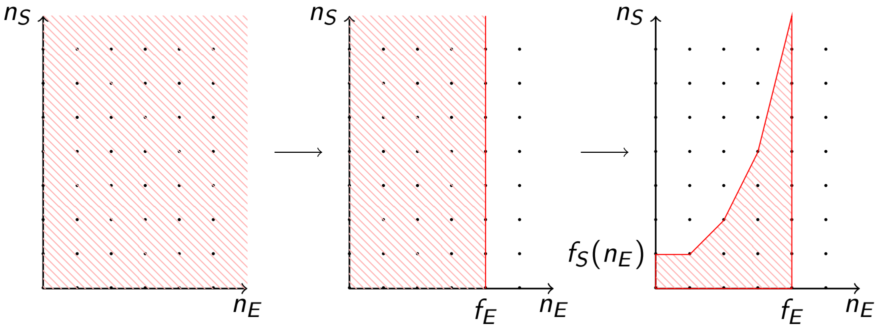
Synthesis problem for $\mathrm{FO}^{\mathrm{pref}}[\lesssim]$



Lemma

Beyond some threshold f_E, if Environment can win with some number of tokens, they can win with a larger number of tokens

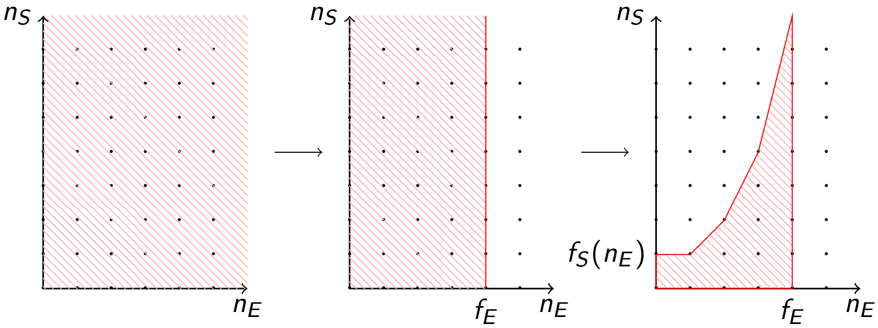
Synthesis problem for $\mathrm{FO}^{\mathrm{pref}}[\lesssim]$



Lemma

There exists $f_S : \mathbb{N} \to \mathbb{N}$ such that for every $n_E \in \mathbb{N}$, if System can win with $> f_S(n_E)$ tokens when Environment has n_E tokens, then System can already win with $f_S(n_E)$ tokens

Synthesis problem for $\mathrm{FO}^{\mathrm{pref}}[\lesssim]$



Lemma

For fixed $n_S, n_E \in \mathbb{N}$, one can decide whether System can win with n_S tokens when Environment has n_E tokens

Conclusion

Logic\Agents	System only	Partitioned	Shared
$FO^2[\sim]$	decidable	decidable	undecidable
$FO[\sim]$	decidable	decidable	undecidable
$FO^{pref}[\lesssim]$	decidable	decidable	undecidable
$FO^2[\sim,<]$	decidable	undecidable	undecidable
$FO^2[\sim,+1]$	decidable	undecidable	undecidable
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Conjecture

The synthesis problem for $\mathsf{FO}^2[\lesssim]$ *with partitioned agents is decidable*

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$FO^2[\sim, <, +1]$	decidable	undecidable	undecidable

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The synthesis problem for $FO^2[\lesssim]$ *with partitioned agents is decidable*

We considered a centralized strategy. What about distributed strategies ?