Proofs and Programs

TD 9 - Let's talk abbout Equality (2)

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Goal Go further in the exploration of the properties of the Identity and equivalence types.

From your lecture Among the many ways to express the idea of equivalence between two types $A, B : \mathcal{U}$, the following have been presented:

i. f has a **quasi-inverse**, if exists $g : B \to A$, $\alpha : g \circ f \sim \mathsf{id}_A$, and $\beta : f \circ g \sim \mathsf{id}_B$:

$$\mathsf{quasi\text{-}inverse}(f) \quad \equiv \quad \sum_{g:B\to A} (g \circ f \sim \mathsf{id}_A) \times (f \circ g \sim \mathsf{id}_B)$$

ii. f is an **equivalence** between A and B, if exists $g : B \to A$ such that $\gamma : f \circ g \sim \mathsf{id}_B$, and $h : B \to A$, such that $\eta : h \circ f \sim \mathsf{id}_A$:

$$\mathsf{is\text{-}equiv}(f) \quad \equiv \quad \left(\sum_{h:B\to A} h \circ f \sim \mathsf{id}_A \right) \times \left(\sum_{g:B\to A} f \circ g \sim \mathsf{id}_B \right).$$

These two presentations have been proven logically equivalent. Then, types A and B are **equivalent**, denoted $A \simeq_{\mathcal{U}} B$, if there is an equivalence between A and B; up to logical equivalence, the formal definition in HoTT has been given:

$$A \simeq_{\mathcal{U}} B \quad \equiv \quad \sum_{f:A\to B} \mathsf{is\text{-}equiv}(f)$$

Exercice 1 (Warming up - Dependent transport)**.** Recall that given a fibration $P : A \to \mathcal{U}$, two elements $x, y : A$ and a path $p : \mathsf{Id}_A(x, y)$, one can define $\mathsf{tr}^P\ \ p : P(x) \to P(y)$ such that, for any $u : P(x)$, $(\mathsf{tr}^P\ \ \mathsf{refl}_x)\ u \equiv u$. Show that

- for $x, y, z : A$, $p : \mathsf{Id}_A(x, y)$, $q : \mathsf{Id}_A(y, z)$ and $u : P(x)$, then $\quad (\mathsf{tr}^P\ \ q)(\mathsf{tr}^P\ \ p\ u) = \mathsf{tr}^P\ \ (p \cdot q)\ u$;
- for $f : B \to A$, $p : \mathsf{Id}_B(b, b')$ and $u : P(f\ b)$, then $\quad \mathsf{tr}^{P\circ f}\ \ p\ u) =_{P(f\ b')} \mathsf{tr}^P\ \ (\mathsf{ap}_f\ \ p)\ u$;
- for $P, Q : A \to \mathcal{U}$, and $h : \prod_{x:A} P(x) \to Q(x)$, $x, x' : A$, $p : \mathsf{Id}_A(x, x')$, and $u : P(x)$, then $\mathsf{tr}^Q\ \ p\ (h\ x\ u) = h\ x'\ (\mathsf{tr}^P\ \ p\ u)$.

Exercice 2 (Is everything refl?)**.** Why do the induction principles for identity types not allow us to construct a function $f : \prod_{x:A} \prod_{p:x=x} (p = \mathsf{refl}_x)$, which satisfies the definitional equality $f(x, \mathsf{refl}_x) \equiv \mathsf{refl}_{\mathsf{refl}_x}$?

Exercice 3 (Type Equivalences on Inductive Types)**.** We are considering the following inductive types: the empty type, $\mathbf{0}$, with no constructors ; the unit type $\mathbf{1}$ with a unic element $\star$; the boolean type $\mathbf{2}$ with two elements true and false ; and the sum type $A + B$ with type constructors $\iota_1 : A \to A + B$, $\iota_2 : B \to A + B$.

1. Recall the dependent induction principles associated with the above types.
2. Prove $\prod x, y : \mathbf{0}.\ \mathsf{Id}_{\mathbf{0}}(x, y) \simeq \mathbf{0}$
3. Prove $\prod x, y : \mathbf{1}.\ \mathsf{Id}_{\mathbf{1}}(x, y) \simeq \mathbf{1}$
4. Prove $\mathbf{2} \simeq\ \ \mathbf{1} + \mathbf{1}$

Exercice 4 (Product extensionality)**.** In this exercise we show that contratry to the arrow type, product type is extensional for the following (intuitive) equivalence: Let $u, v : A \times B$ be pairs, we define their property of being equivalent by the type

$$u \sim_{A\times B} v \equiv \mathsf{Id}_A(\pi_1(u), \pi_1(v)) \times \mathsf{Id}_B(\pi_2(u), \pi_2(v))$$

1. Check that $\mathsf{Id}_{A\times B}(u, v) \to u \sim_{A\times B} v$
2. What about the contraposite?

Exercice 5 (Equivalence lifting)**.** In HoTT, proofs can become involved.. Let us consider the following result, which was part of 2017 final exam. The goal is to prove that if $f : A \to B$ is an equivalence between A and B, then for each pair of elements $a, a' : A$, the map $\mathsf{ap}_{f,a,a'} : a =_A a' \to f(a) =_B f(a')$ is an equivalence as well. In the sequell we will left the subscript a, a' in ap_f implicit.

Question 1 As a quasi-inverse candidate for ap_f, let us consider $G(\cdot)$, defined by

$$G(q) \equiv \alpha(a)^{-1} \cdot \mathsf{ap}_g(q) \cdot \alpha(a') \tag{1}$$

To satisfy the requirement, we have to exhibit homotopies γ (as left inverse) and δ (as right inverse):

$$\gamma : \prod_{p:J} G(\mathsf{ap}_f(p)) =_J p \quad \text{et} \quad \delta : \prod_{q:K} \mathsf{ap}_f(G(q)) =_K q$$

a) What are the types J and K? What is the type of the candidate $G(\cdot)$?

b) Prove the existence of a witness γ.

c) Why is that not possible to use a similar approach to prove the existence of δ?

Question 2. Let $T : \mathcal{U}$ and $\varphi : T \to T$ such that $\varepsilon : \prod_{x:T} \varphi(x) =_T \mathsf{id}_T(x)$.

a) Given $x, x' : T$ and $r : x =_T x'$, prove that $\varepsilon(x)^{-1} \cdot \mathsf{ap}_\varphi(r) \cdot \varepsilon(x') =_S r$, where the type S will be made explicit.

b) Conclude that, for all $x : T$, $\varepsilon(\varphi(x)) = \mathsf{ap}_\varphi(\varepsilon(x))$.

Question 3. Given $x : A$, let us note $\nu(x) \equiv \beta(f(x))^{-1} \cdot \beta(f(x))$.

a) State the type of $\nu(\cdot)$.

b) Prove that $\beta(f(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_g(q)) \cdot \beta(f(a')) =_K q$.

c) Simplify the path $\nu(a) \cdot \mathsf{ap}_f(G(q)) \cdot \nu(a')$.

d) Conclude for the existence of an homotopy proof δ such that

$$\delta \quad : \quad \prod_{q:K} \mathsf{ap}_f(G(q)) =_K q$$

Exercice 6 (More on equivalences)**.** Prove the following statements:

1. (identity) For all $A : \mathcal{U}$, $\mathsf{quasi\text{-}inverse}\,(\mathsf{id}_A)$;
2. (between identity types) For all $A : \mathcal{U}$, $x, y : A$ and $p : x =_A y$,
   - $(p \cdot -) : y = z \to x = z$ and $(p^{-1} \cdot -)$ are quasi-inverse one of the other ;
   - $(- \cdot p) : z = x \to z = y$ et $(- \cdot p^{-1})$ are quasi-inverse one of the other.
3. (transport) If $P : A \to \mathcal{U}$, then $\mathsf{tr}^P(p, -) : P(x) \to P(y)$ has $\mathsf{tr}^P(p^{-1}, -)$ for a quasi-inverse.