# Proofs and Programs

## TD 8 - Let's talk abbout Equality

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29 March 2018

**Goal** Go further in the exploration of the properties of the Identity type.

**From your lecture** The homotopy type theory is based on Per Martin Löf Intuitionist Type theory (seen last week) but put the emphasis on studying *propositionnal* equality by the mean of a particular inductive type, the *Identity type* $\mathsf{Id}_A(a,b)$ defined for any type $A$ and elements $a, b : A$ by:

$$(\mathsf{Id}-type)\ \frac{A:\mathcal{U} \qquad a:A \quad b:A}{\mathsf{Id}_A(a,b):\mathcal{U}} \qquad (\mathsf{Id}-intro)\ \frac{a:A}{\mathsf{refl}_a:\mathsf{Id}_A(a,a)}$$

$$(\mathsf{Id}-elim)\ \frac{x:A,y:A,z:\mathsf{Id}_A(x,y)\vdash C(x,y,z):\mathcal{U} \qquad x:A\vdash c:C\langle x,x,\mathsf{refl}_x/x,y,z\rangle \qquad p\ :\ \mathsf{Id}_A(a,b)}{\mathbf{J}[x.y.z.C]\ p\ (\lambda x.c):C\langle a,b,p/x,y,z\rangle}$$

$$\beta\text{-rule}:\quad \mathbf{J}[x.y.z.C]\ \ \mathsf{refl}_a\ (\lambda x.c)\equiv\ c\langle x/a\rangle(andisoftypeC\langle a,a,\mathsf{refl}_a/x,y,z\rangle)$$

In class, you have seen that $Id_A$ defines an equivalence relation over elements of type $A$ and that it has a structure of groupoid. In particular, there is a concatenation operation $\cdot$ (transitivity) and an inverse operator $\_^{-1}$ (symmetry) such that:

- $(\mathsf{refl}_x)^{-1}\equiv\mathsf{refl}_x$ and $\mathsf{refl}_x\cdot\mathsf{refl}_x\equiv\mathsf{refl}_x$,
- (unit) for all $p:\mathsf{Id}_A(a,b)$, $\quad p\cdot\mathsf{refl}_b=p=\mathsf{refl}_a\cdot p$
- (inverse) for all $p:\mathsf{Id}_A(a,b)$, $\quad p\cdot p^{-1}=\mathsf{refl}_a$ and $p^{-1}\cdot p=\mathsf{refl}_b$
- (associativity) for all $p:\mathsf{Id}_A(a,b),q:\mathsf{Id}_A(b,c),r:\mathsf{Id}_A(c,d)$, $\quad (p\cdot q)\cdot r=p\cdot(q\cdot r)$

In the above, $p_1=p_2$ is a shortcuts for the propositional equality $\mathsf{Id}_{\mathsf{Id}_A(a,b)}(p_1,p_2)$ between two paths $p_1,p_2:\mathsf{Id}_A(a,b)$. Proving this equalities means there is a term inhabiting these types.

Lastly, you have seen that simple functions preserve the groupoid structure with the *transport* properties:

- (transport) for all $f:A\to B$, $a,b:A$ there is $\mathsf{ap}_f:\mathsf{Id}_A(a,b)\to\mathsf{Id}_B(f\ a,f\ b))$, such that for $f:A\to A$, $x:A$, $\mathsf{ap}_f\ \ \mathsf{refl}_x\equiv\mathsf{refl}_{f\ x}$

Moreover the following properties hold for $f:A\to B$, $g:B\to C$, $p\,\mathsf{Id}_A(x,y)$, $q:\mathsf{Id}_A(y,z)$:

1. $\mathsf{ap}_f(p^{-1})=_{f(y)=f(x)}\mathsf{ap}_f(p)^{-1}$
2. $\mathsf{ap}_f(p\cdot q)=_{f(x)=f(z)}\mathsf{ap}_f(p)\cdot\mathsf{ap}_f(q)$
3. $\mathsf{ap}_g(\mathsf{ap}_f(p))=_{g\circ f(x)=g\circ f(y)}\mathsf{ap}_{g\circ f}(p)$;
4. $\mathsf{ap}_{\lambda x^A.x}(p)=_{x=y}p$

**Exercice 1** (Warming up)**.** Show that

1. if $\mathsf{Id}_{A\to B}(f,g)$ and $\mathsf{Id}_A(x,y)$, then $\mathsf{Id}_B(f(x),g(y))$.
2. one of the four properties of $\mathsf{ap}$.

**Exercice 2** (Homotopies)**.** Given $f, g : A \to B$, define the new type $f \sim_{A\to B} g$ ($f \sim g$ for short)

$$f \sim_{A\to B} g \quad \equiv \quad \prod_{x:A} \mathsf{Id}_B\,(f\ x, g\ x)$$

A witness $\eta : f \sim_{A\to B} g$ is named a *path homotopy* from $f$ to $g$. Given $f, g : A \to B$, such that $\eta : f \sim g$, show that

1. (commutation) for all path $p : \mathsf{Id}_A(x, x')$, $\quad \eta(x) \cdot \mathsf{ap}_g(p) =_{f(x)=g(x')} \mathsf{ap}_f(p) \cdot \eta(x')$
2. (equivalence) the relation $\sim_{A\to B}$ is an equivalence relation over $A \to B$.
3. ($\mathsf{happly}$) $(\Pi f, g : A \to B)\, \mathsf{Id}_{A\to B}(f, g) \to f \sim_{A\to B} g$. What about the contraposit?

**Exercice 3** (Dependent transport)**.** Given a fibration $P : A \to \mathcal{U}$, a path $p : \mathsf{Id}_A(x, y)$, and a section $f : \prod_{x:A} P(x)$, show that:

- there exists $\mathsf{tr}^P\ \ p : P(x) \to P(y)$ such that, for any $u : P(x)$, $(\mathsf{tr}^P\ \ \mathsf{refl}_x)\ u \equiv_{P(x)} u$,
- there exists $\mathsf{apd}_f : \prod_{p:x=x'} (\mathsf{tr}^P\ \ p)(f\ x) =_{P(x')} f\ x'$, such that $\mathsf{apd}_f\,(\mathsf{refl}_x) =_{P(x)} \mathsf{refl}_{f(x)}$,
- for $x, y, z : A$, $p : \mathsf{Id}_A(x, y)$, $q : \mathsf{Id}_A(y, z)$ and $u : P(x)$ $(\mathsf{tr}^P\ \ q)((\mathsf{tr}^P\ \ p)\ u) = (\mathsf{tr}^P\ \ (p \cdot q))\ u$
- for $f : B \to A$, $p : \mathsf{Id}_A(a, a')$ and $u : P(f(a))$, then $\mathsf{tr}^{P\circ f}(p, u) =_{P(f(a'))} \mathsf{tr}^P \left(\mathsf{ap}_f(p), u\right)$
- for $P, Q : A \to \mathcal{U}$, and $h : \prod_{x:A} P(x) \to Q(x)$, $x, x' : A$, $p : \mathsf{Id}_A(x, x')$, and $u : P(x)$, then $\mathsf{tr}^Q(p, f(x, u)) = f\left(x', \mathsf{tr}^P(p, u)\right)$.

**Exercice 4** (Is everything $\mathsf{refl}$?)**.** Why do the induction principles for identity types not allow us to construct a function $f : \prod_{x:A} \prod_{p:x=x} (p = \mathsf{refl}_x)$ with the defining equation $f(x, \mathsf{refl}_x) \equiv \mathsf{refl}_{\mathsf{refl}_x}$?

**Exercice 5** (Product extensionality)**.** In this exercise we show that contratry to the arrow type, product type is extensional for the following (intuitive) equivalence: Let $u, v : A \times B$ be pairs, we define their property of being equivalent by the type

$$u \sim_{A\times B} v \equiv \mathsf{Id}_A(\pi_1(u), \pi_1(v)) \times \mathsf{Id}_B(\pi_2(u), \pi_2(v))$$

1. Check that $\mathsf{Id}_{A\times B}(u, v) \to u \sim_{A\times B} v$
2. What about the contraposite?