Proofs and Programs

Week 2, TD 2 - Natural deduction

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N.B. In the following we do not detail every questions. Please only refer to full answers to know which level of details is expected from you. Feel free to contact your teachers for any further questions.

Exercice 1 (Warm up!)**.** Done in class.

Exercice 2 (Toolbox)**.** a) If (R) is derivable then there exists a derivation tree T for (R) (ie from the premises of (R) to its conclusion). Thus, if there exists a derivation for every premises of (R) one can build a derivation of its conclusion by plugging each of the premises' derivations above their corresponding premises in T. So (R) is admissible.

b) Right implication is trivial since a derivation in NJ is also a derivation in NJ+(R).

Let us show the left implication by induction on the derivation $\dfrac{\Pi}{\Delta \vdash_{\mathrm{NJ}+R} A}\ (X)$:

- base cases : if $(X) = (\mathrm{Hyp})$ or $(\top_I)$ then the derivation is also a derivation in NJ as (X) is in NJ.
- induction cases : if $(X) = (\mathrm{R})$, then all its premises have a smaller derivation in NJ + (R) so by induction hypothesis all its premises have a derivation in NJ. But (R) is admissible so this implies that its conclusion also have a derivation in (NJ). Finally, if (X) is any other rules left in NJ then applying the induction hypothesis on the premises of (X) leads to a derivation of $\Delta \vdash A$ in NJ.

c) Done in class

d) **HW**

Exercice 3 (Contraction and weakening)**.** State precisely, then prove by induction the following (informal) statements:

(Weakening) Let us prove the admissibility of the weakening rule $\dfrac{\Delta \vdash B}{\Delta, A \vdash B}\ (Wk)$ by induction on the derivation $\dfrac{\Pi}{\Delta \vdash_{\mathrm{NJ}} B}\ (X)$:

- base cases : if $(X) = (\mathrm{Hyp})$ then $B \in \Delta$ so $\dfrac{B \in \Delta \cup \{A\}}{\Delta, A \vdash B}$ (Hyp) is a derivation for $\Delta, A \vdash B$ in NJ. And similarly for $(X) = (\top_I)$.
- induction cases : if $(X) = (\mathrm{R})$, then all its premises have a smaller derivation in NJ + (R) so by induction hypothesis all its premises have a derivation in NJ. But (R) is admissible so this implies that its conclusion also have a derivation in (NJ). Finally, if (X) is any other rules left in NJ then applying the induction hypothesis on its premises leads to a derivation of $\Delta, A \vdash B$ in NJ as every rules in NJ is context preserving.

(Contraction) HW

Exercice 4 (There is no *alternative fact*...)**.** Done in class.

Exercice 5 (Stability and decidability)**.** Note that if A is decidable then it is stable. Indeed

$$\dfrac{\dfrac{}{A, \neg\neg A \vdash A}\ (Hyp) \qquad \dfrac{\dfrac{\dfrac{}{\neg A, \neg\neg A \vdash \neg A \Rightarrow \bot}\ (Hyp) \qquad \dfrac{}{\neg A, \neg\neg A \vdash \neg A}\ (Hyp)}{\neg A, \neg\neg A \vdash \bot}\ (\Rightarrow_E)}{\neg A, \neg\neg A \vdash A}\ (\bot_E)}{\dfrac{A \vee \neg A, \neg\neg A \vdash A}{\vdash (A \vee \neg A) \Rightarrow (\neg\neg A \Rightarrow A)}\ (\Rightarrow_I\ (\times 2))}\ (\vee_L)$$

However, the converse implication is false in general.

a) $\bot$ and $\top$ are both decidable (hence stable) indeed:

$$\dfrac{\dfrac{\dfrac{}{\bot \vdash \bot}\ (Hyp)}{\vdash \neg\bot}\ (\Rightarrow_I)}{\vdash \bot \vee \neg\bot}\ (\vee_I) \qquad \dfrac{\dfrac{}{\vdash \top}\ (\top_I)}{\vdash \top \vee \neg\top}\ (\vee_I)$$

b) In general $\neg A$ is not decidable in NJ, however it is stable, cf ex. 4.c)!

c) **HW**

d) **HW**

Exercice 6 (De Morgan laws)**.** The first De Morgan equivalence was proved in class. However, the second De Morgan equivalence does not hold in NJ since the right implication cannot be derived. If it was derivable then its derivation would start by

$$\dfrac{\dfrac{\dfrac{\neg(A \wedge B), A \vdash \bot}{\neg(A \wedge B) \vdash \neg A}\ (\Rightarrow_I)}{\neg(A \wedge B) \vdash \neg A \vee \neg B}\ (\vee_I)}{\vdash \neg(A \wedge B) \Rightarrow \neg A \vee \neg B}\ (\Rightarrow_I)$$

but then, no rules in NJ allow to go further in the derivation (and it would be the same if we chose B instead of A).

On the other side the left implication is still derivable:

$$\dfrac{\dfrac{\dfrac{}{\neg A, A \wedge B \vdash \bot}\ (\pi_A) \qquad \dfrac{}{\neg B, A \wedge B \vdash \bot}\ (\pi_B)}{\neg A \vee \neg B, A \wedge B \vdash \bot}\ (\vee_L)}{\vdash \neg A \vee \neg B \Rightarrow \neg(A \wedge B)}\ (\Rightarrow_I\ (\times 2))$$

with

$$\pi_A = \dfrac{\dfrac{\dfrac{}{\neg A, A, B \vdash A \Rightarrow \bot}\ (Hyp) \qquad \dfrac{}{\neg A, A, B \vdash A}\ (Hyp)}{\neg A, A, B \vdash \bot}\ (\Rightarrow_E)}{\neg A, A \wedge B \vdash \bot}\ (\wedge_L)$$

and

$$\pi_B = \dfrac{\dfrac{\dfrac{}{\neg B, A, B \vdash B \Rightarrow \bot}\ (Hyp) \qquad \dfrac{}{\neg B, A, B \vdash B}\ (Hyp)}{\neg B, A, B \vdash \bot}\ (\Rightarrow_E)}{\neg B, A \wedge B \vdash \bot}\ (\wedge_L)$$

Exercice 7 (A logic koan: Master Foo and the excluded middle)**. HW**

Exercice 8. (Towards Glivenko's theorem) **HW**