Proofs and Programs

TD 2 - Natural deduction

Philippe Audebaud, Aurore Alcolei

7 February 2017 — **HW** due 13 Feb., 8h **dead line**

Highlights-

- be familiar with proof derivation in intuitionistic natural deduction (NJ, see appendix) on paper and in the COQ proof assistant;
- investigate the expressivity of NJ;
- play with basics concepts in proof theory.

Rules and definitions for Natural deduction are recalled in the appendix. From the intuitionistic point of view, negation is not a proposition constructor: notation (*syntactic sugar*) $\neg A$ stands for $A \Rightarrow \bot$.

Exercice 1 (Warm up!)**.** On paper, then in COQ, carrefully build the derivation trees for

a) $\vdash_{\mathrm{NJ}} A \Rightarrow A$;

b) $\vdash_{\mathrm{NJ}} (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$;

c) $\vdash_{\mathrm{NJ}} (A \wedge B) \Rightarrow (A \vee B)$.

Exercice 2 (Toolbox)**.** A rule (R) is **derivable** (in NJ) if its conclusion can be derived from its premises using (NJ) inference rules. A rule (R) is **admissible** (in NJ) if its conclusion is derivable whenever all its premises are derivable (in the same inference system).

a) Prove that any derivable rule is admissible.

b) Let (R) be some admissible rule wrt to NJ. Prove that for any judgment $\Delta \vdash A$,

$$\Delta \vdash_{\mathrm{NJ}} A \quad \text{iff} \quad \Delta \vdash_{\mathrm{NJ}+R} A$$

Hint : one way or the other you may need to reason by induction over the proof tree.

You have just proved that adding an admissible rule to a system does not change its *expressive power*! You are now allowed to use in your derivation trees every rules that is proved to admissible in NJ.

c) Prove that the following rules are derivable:

$$(R_1)\ \frac{\Delta, A \Rightarrow B \vdash A}{\Delta, A \Rightarrow B \vdash B} \qquad (R_2)\ \frac{\Delta, \neg A \vdash A}{\Delta, \neg A \vdash \bot} \qquad \textbf{HW} \quad (R_3)\ \frac{\Delta, A \vdash \neg A}{\Delta, A \vdash \bot}$$

d) Prove that the following rules are admissible (you can assume admissibility of weakening - Exercise 3):

$$(\mathrm{Cut})\ \frac{\Delta, A \vdash B \quad \Delta \vdash A}{\Delta \vdash B} \qquad (\wedge_L)\ \frac{\Delta, A, B \vdash C}{\Delta, A \wedge B \vdash C} \qquad (\vee_L)\ \frac{\Delta, A \vdash C \quad \Delta, B \vdash C}{\Delta, A \vee B \vdash C}$$

$$\textbf{HW} \quad (\Rightarrow_L)\ \frac{\Delta, A, B \vdash C}{\Delta, A, A \Rightarrow B \vdash C}$$

Exercice 3 (Contraction and weakening)**.** Using the wording from Exercise 2, state precisely, then prove by induction the following (informal) statements:

(Weakening) If $\Delta \vdash_{\mathrm{NJ}} B$, then $\Delta, A \vdash_{\mathrm{NJ}} B$;

(Contraction) **HW** If $\Delta, A, A \vdash B$, then $\Delta, A \vdash B$.

Exercice 4 (There is no *alternative fact...*)**.** In COQ, prove the judgments:

a) $\vdash_{\mathrm{NJ}} A \Rightarrow \neg\neg A$;

b) $\vdash_{\mathrm{NJ}} \neg(A \wedge \neg A)$;

c) $\vdash_{\mathrm{NJ}} \neg\neg\neg A \Rightarrow \neg A$.

Exercice 5 (De Morgan laws)**.** As usual, equivalence is syntactic sugar: $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$. For the following well-known equivalences, decide for each side if the judgment holds in NJ, or provide an informal argument that refute this judgment:

$$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B \qquad \neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$

Exercice 6 (Stability and decidability)**.** We say that a proposition is ***stable*** when $\vdash_{NJ} \neg\neg A \Rightarrow A$; it is ***decidable*** when $\vdash_{NJ} A \vee \neg A$. The following properties amount to analysing the action of $\neg\neg(\cdot)$ as a closure operator on propositions:

a) For $A \in \{\bot, \top\}$, check whether A is stable (resp. decidable);

b) For any proposition A, check whether $\neg A$ is stable (resp. decidable);

c) Show the the set of stable propositions is (algebrically) stable under $\Rightarrow$ and $\wedge$ (operations).

d) Prove the existence of an operator $F(\cdot)$ such that, for any proposition A, $F(A)$ stable implies A decidable.

Exercice 7 (A logic koan: Master Foo and the excluded middle)**.** **HW** Listen to this story:

Master Foo once was visited by a student in logic asking him about the true meaning of the excluded middle:

"*How can I know whether something is true or false? I do not know whether I will die an old man?*". Master Foo smiled and made the following offer:

"*I will choose between giving you 1.000 bananas so you shall never go hungry, or if you gave me that many bananas, I will grant you any wish.*"

The student accepts and Master Foo opts for the latter. The student wishes for a complete and thorough understanding of logic. Master Foo accepts and lets him wander in the depth of the forest. Later he comes back with the bananas he had to steal from a local gang of monkeys. Master Foo replies:

"*That is very nice, thank you. Did I tell you that while you were gone I changed my mind about the offer?*"

and hands him his bananas back. Upon receiving the bananas, the student was enlightened.

Moral of the story : prove the following judgement:

$$\vdash_{\mathrm{NJ}} \neg\neg(A \vee \neg A)$$

Exercice 8. (Towards Glivenko's theorem) **HW** In the same line of exercises 4 and 7 prove the following statements involving double negation in COQ.

a) $\vdash_{\mathrm{NJ}} \neg\neg P \Rightarrow \neg\neg(P \vee Q)$;

b) $\vdash_{\mathrm{NJ}} (P \Rightarrow \neg\neg Q) \Rightarrow \neg\neg(P \Rightarrow Q)$;

c) $\vdash_{\mathrm{NJ}} (\neg\neg P \wedge \neg\neg Q) \Rightarrow \neg\neg(P \wedge Q)$.

A Short reminder on Intuitionistic Natural Deduction (NJ)

As far as definitions or notations are concerned, always refer to Lecture home page.

Formulas A,B, ... are **propositions**, inductively defined by:

$$A, B, \ldots \quad ::= \quad X \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \Rightarrow B$$

where $X, Y, \ldots$ range over a given (denumerable) set of **propositional variables**.

A **context** is a multiset of propositions (repetition is allowed and the order is not relevant). A **judgement** is a pair denoted $\Delta \vdash A$, where Δ is a context and A is a proposition. "$A \in \Delta$" means the proposition A occurs in the list (of propositions) Δ.

Deduction rules for Intuitionistic Natural Deduction (NJ) are the following:

$$\frac{A \in \Delta}{\Delta \vdash A}\ (\text{Hyp}) \qquad \frac{\Delta \vdash \bot}{\Delta \vdash A}\ (\bot_E) \qquad \frac{}{\Delta \vdash \top}\ (\top_I)$$

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \Rightarrow B}\ (\Rightarrow_I) \qquad \frac{\Delta \vdash A \Rightarrow B \quad \Delta \vdash A}{\Delta \vdash B}\ (\Rightarrow_E)$$

$$\frac{\Delta \vdash A \quad \Delta \vdash B}{\Delta \vdash A \wedge B}\ (\wedge_I) \qquad \frac{\Delta \vdash A \wedge B}{\Delta \vdash A}\ (\wedge_{E(L)}) \qquad \frac{\Delta \vdash A \wedge B}{\Delta \vdash B}\ (\wedge_{E(R)})$$

$$\frac{\Delta \vdash A}{\Delta \vdash A \vee B}\ (\vee_{I(L)}) \qquad \frac{\Delta \vdash B}{\Delta \vdash A \vee B}\ (\vee_{I(R)}) \qquad \frac{\Delta \vdash A \vee B \quad \Delta, A \vdash C \quad \Delta, B \vdash C}{\Delta \vdash C}\ (\vee_E)$$

We note $\Delta \vdash_{\text{NJ}} A$ whenever there exists a derivation tree Π by these rules, such that $\dfrac{\Pi}{\Delta \vdash A}\ (X)$, where X is the name of the last inference rule involved. We say that NJ **proves** A whenever $\vdash_{\text{NJ}} A$ is derivable (empty context).